Uncertainties and Error Propagation

By Vern Lindberg

1. Systematic and random errors.

No measurement made is ever exact. The **accuracy** (correctness) and **precision** (number of significant figures) of a measurement are always limited by the degree of refinement of the apparatus used, by the skill of the observer, and by the basic physics in the experiment. In doing experiments we are trying to establish the best values for certain quantities, or trying to validate a theory. We must also give a **range of possible true values** based on our limited number of measurements.

Why should repeated measurements of a single quantity give different values? Mistakes on the part of the experimenter are possible, but we do not include these in our discussion. A careful researcher should not make mistakes! (Or at least she or he should recognize them and correct the mistakes.)

We use the synonymous terms **uncertainty, error, or deviation** to represent the variation in measured data. Two types of errors are possible. **Systematic error** is the result of a mis-calibrated device, or a measuring technique which always makes the measured value larger (or smaller) than the "true" value. An example would be using a steel ruler at liquid nitrogen temperature to measure the length of a rod. The ruler will contract at low temperatures and therefore overestimate the true length. Careful design of an experiment will allow us to eliminate or to correct for systematic errors.

Even when systematic errors are eliminated there will remain a second type of variation in measured values of a single quantity. These remaining deviations will be classed as **random errors**, and can be dealt with in a statistical manner. This document does not teach statistics in any formal sense, but it should help you to develop a working methodology for treating errors.

2. Determining random errors.

How can we estimate the uncertainty of a measured quantity? Several approaches can be used, depending on the application.

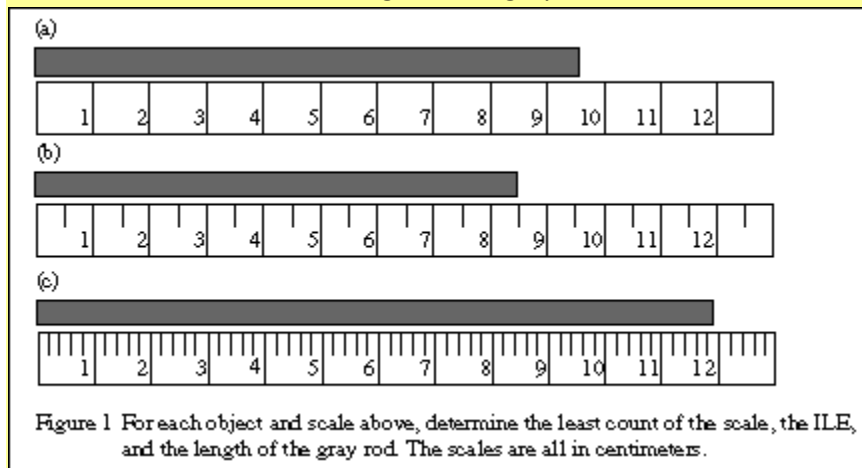### (a) Instrument Limit of Error (ILE) and Least Count

The **least count** is the smallest division that is marked on the instrument. Thus a meter stick will have a least count of 1.0 mm, a digital stop watch might have a least count of 0.01 sec.

The **instrument limit of error**, **ILE** for short, is the precision to which a measuring device can be read, and is always equal to or smaller than the least count. Very good measuring tools are calibrated against standards maintained by the National Institute of Standards and Technology.

The Instrument Limit of Error is generally taken to be the least count or some fraction (1/2, 1/5, 1/10) of the least count). You may wonder **which to choose**, the least count or half the least count, or something else. No hard and fast rules are possible, instead you must be guided by common sense. If the space between the scale divisions is large, you may be comfortable in estimating to 1/5 or 1/10 of the least count. If the scale divisions are closer together, you may only be able to estimate to the nearest 1/2 of the least count, and if the scale divisions are very close you may only be able to estimate to the least count.

For some devices the ILE is given as a tolerance or a percentage. Resistors may be specified as having a tolerance of 5%, meaning that the ILE is 5% of the resistor's value.

Problem: For each of the following scales (all in centimeters) determine the least count, the ILE, and read the length of the gray rod.

Figure 1 For each object and scale above, determine the least count of the scale, the ILE, and the length of the gray rod. The scales are all in centimeters.

Answer

	Least Count (cm)	ILE (cm)	Length (cm)
(a)	1	0.2	9.6
(b)	0.5	0.1	8.5
(c)	0.2	0.05	11.90

(b) Estimated Uncertainty

Often other uncertainties are larger than the ILE. We may try to balance a simple beam balance with masses that have an ILE of 0.01 grams, but find that we can vary the mass on one pan by as much as 3 grams without seeing a change in the indicator. We would use half of this as the **estimated uncertainty**, thus getting uncertainty of ±1.5 grams.

Another good example is determining the focal length of a lens by measuring the distance from the lens to the screen. The ILE may be 0.1 cm, however the depth of field may be such that the image remains in focus while we move the screen by 1.6 cm. In this case the estimated uncertainty would be half the range or ±0.8 cm.

Problem: I measure your height while you are standing by using a tape measure with ILE of 0.5 mm. Estimate the uncertainty. Include the effects of not knowing whether you are "standing straight" or slouching.

Solution.

There are many possible correct answers to this. However the answer

$\Delta h = 0.5$ mm is certainly **<u>wrong</u>** Here are some of the problems in measuring.

1. As you stand, your height keeps changing. You breath in and out, shift from one leg to another, stand straight or slouch, etc. I bet this would make your height uncertain to at least 1.0 cm.
2. Even if you do stand straight, and don't breath, I will have difficulty measuring your height. The top of your head will be some horizontal distance from the tape measure, making it hard to measure your height. I could put a book on your head, but then I need to determine if the book is level.

I would put an uncertainty of **<u>1 cm</u>** for a measurement of your height.

(c) Average Deviation: Estimated Uncertainty by Repeated Measurements

The statistical method for finding a value with its uncertainty is to repeat the measurement several times, find the **<u>average</u>**, and find either the **<u>average deviation</u>** or the **<u>standard deviation</u>**.

Suppose we repeat a measurement several times and record the different values. We can then find the **average** value, here denoted by a symbol between angle brackets, <t>, and use it as our best estimate of the reading. How can we determine the uncertainty? Let us use the following data as an example. Column 1 shows a time in seconds.

Table 1. Values showing the determination of average, average deviation, and standard deviation in a measurement of time. Notice that to get a non-zero average deviation we

must take the absolute value of the deviation.			
Time, t, sec.	(t - <t>), sec	|t - <t>|, sec	$\left(t - <t>\right)^2$
7.4	-0.2	0.2	0.04
8.1	0.5	0.5	0.25
7.9	0.3	0.3	0.09
7.0	-0.6	0.6	0.36
<t> = 7.6	<t-<t>>= 0.0	<|t-<t>|>= 0.4	$\left\langle\left(t - <t>\right)^2\right\rangle$ = 0.247 Std. dev = 0.50

A simple **average** of the times is the sum of all values (7.4+8.1+7.9+7.0) divided by the number of readings (4), which is 7.6 sec. We will use angular brackets around a symbol to indicate average; an alternate notation uses a bar is placed over the symbol.

Column 2 of Table 1 shows the deviation of each time from the average, (t-<t>). A simple average of these is zero, and does not give any new information.

To get a non-zero estimate of deviation we take the average of the absolute values of the deviations, as shown in Column 3 of Table 1. We will call this the **average deviation, Δt**.

Column 4 has the squares of the deviations from Column 2, making the answers all positive. The sum of the squares is divided by 3, (one less than the number of readings), and the square root is taken to produce the sample **standard deviation**. An explanation of why we divide by (N-1) rather than N is found in any statistics text. The sample standard deviation is slightly different than the average deviation, but either one gives a measure of the variation in the data.

If you use a spreadsheet such as Excel there are built-in functions that help you to find these quantities. These are the Excel functions.

=SUM(A2:A5)   Find the sum of values in the range of cells A2 to A5.

=COUNT(A2:A5)   Count the number of numbers in the range of cells A2 to A5.

=AVERAGE(A2:A5)	Find the average of the numbers in the range of cells A2 to A5.
=AVEDEV(A2:A5)	Find the average deviation of the numbers in the range of cells A2 to A5.
=STDEV(A2:A5)	Find the sample standard deviation of the numbers in the range of cells A2 to A5.

For a second example, consider a measurement of length shown in Table 2. The average and average deviation are shown at the bottom of the table.

Table 2. Example of finding an average length and an average deviation in length. The values in the table have an **excess of significant figures**. Results should be rounded and can be reported as (15.5 ± 0.1) m or (15.47 ± 0.13) m. If you use standard deviation the length is (15.5 ± 0.2) m or (15.47 ± 0.18) m.

Length, x, m	$\lvert x - <x> \rvert$, m	$\left(x - <x>\right)^2$
15.4	0.06667	0.004445
15.2	0.26667	0.071112
15.6	0.13333	0.017777
15.7	0.23333	0.054443
15.5	0.03333	0.001111
15.4	0.06667	0.004445
Average 15.46667 m	±0.133333 m	St. dev. ±0.17512

We round the uncertainty to one or two significant figures (more on rounding in Section 7), and round the average to the same number of digits relative to the decimal point. Thus the average length with average deviation is either (15.47 ± 0.13) m or (15.5 ± 0.1) m. If we use standard deviation we report the average length as (15.47±0.18) m or (15.5±0.2) m.

Follow your instructor's instructions on whether to use average or standard deviation in your reports.

Problem Find the average, and average deviation for the following data on the length of a pen, L. We have 5 measurements,
(12.2, 12.5, 11.9,12.3, 12.2) cm.

Solution

We have 5 measurements,
(12.2, 12.5, 11.9,12.3, 12.2) cm.

Length (cm)	$\left\| L - L_{ave} \right\|$	$\left(L - L_{ave} \right)^2$
12.2	0.02	0.0004
12.5	0.28	0.0784
11.9	0.32	0.1024
12.3	0.08	0.0064
12.2	0.02	0.0004
Sum 61.1	Sum 0.72	Sum 0.1880
Average 61.1/5 = 12.22	Average 0.14	$\sigma = \sqrt{\dfrac{0.1880}{4}} = 0.22$

To get the average sum the values and divide by the number of measurements.
To get the average deviation,

1. Find the deviations, the absolute values of the quantity (value minus the average), $\left\| L - L_{ave} \right\|$
2. Sum the absolute deviations,
3. Get the average absolute deviation by dividing by the number of measurements

To get the standard deviation

1. Find the deviations and square them

2. Sum the squares
3. Divide by (N-1), the number of measurements minus 1 (here it is 4)
4. Take the square root.

The pen has a length of (12.22 +/- 0.14) cm or (12.2 +/- 0.1) cm [using average deviations] or

(12.22 +/- 0.22) cm or (12.2 +/- 0.2) cm [using standard deviations].

Problem: Find the average and the average deviation of the following measurements of a mass.

(4.32, 4.35, 4.31, 4.36, 4.37, 4.34) grams.

[Solution](#)

Mass (grams)	$\left\lvert m - m_{ave} \right\rvert$	$\left(m - m_{ave} \right)^2$
4.32	0.0217	0.000471
4.35	0.0083	0.000069
4.31	0.0317	0.001005
4.36	0.0183	0.000335
4.37	0.0283	0.000801
4.34	0.0017	0.000003
Sum 26.05	0.1100	0.002684
Average 4.3417	Average 0.022	$\sigma = 0.023$

The same rules as Example 1 are applied. This time there are N = 6 measurements, so for the standard deviation we divide by (N-1) = 5.

The mass is (4.342 +/- 0.022) g or (4.34 +/- 0.02) g [using average

(d) Conflicts in the above

In some cases we will get an ILE, an estimated uncertainty, and an average deviation and we will find different values for each of these. We will be pessimistic and **take the largest of the three values as our uncertainty**. [When you take a statistics course you should learn a more correct approach involving adding the variances.] For example we might measure a mass required to produce standing waves in a string with an ILE of 0.01 grams and an estimated uncertainty of 2 grams. We use 2 grams as our uncertainty.

The proper way to write the answer is

1. Choose the largest of (i) ILE, (ii) estimated uncertainty, and (iii) average or standard deviation.
2. Round off the uncertainty to 1 or 2 significant figures.
3. Round off the answer so it has the same number of digits before or after the decimal point as the answer.
4. Put the answer and its uncertainty in parentheses, then put the power of 10 and unit outside the parentheses.

Problem: I make several measurements on the mass of an object. The balance has an ILE of 0.02 grams. The average mass is 12.14286 grams, the average deviation is 0.07313 grams. What is the correct way to write the mass of the object including its uncertainty? **What is the mistake in each incorrect one?**

1. 12.14286 g
2. (12.14 ± 0.02) g
3. 12.14286 g ± 0.07313
4. 12.143 ± 0.073 g
5. (12.143 ± 0.073) g
6. (12.14 ± 0.07)
7. (12.1 ± 0.1) g
8. 12.14 g ± 0.07 g
9. (12.14 ± 0.07) g

Answer

| 1. 12.14286 g | Way wrong! You need the uncertainty reported with the |

	answer. Also the answer has not been properly rounded off.
2. (12.14 ± 0.02) g	Way wrong! You could not read my writing perhaps. The uncertainty is 0.07 grams. Otherwise the format of the answer is fine.
3. 12.14286 g $\pm$ 0.07313	Way wrong! You need to round off the uncertainty and the answer. Also the answer should be presented within parentheses.
4. $12.143 \pm$ 0.073 g	Almost there. Put parentheses around the numbers and it would be OK. Rounding off one more place is better.
5. $(12.143 \pm$ $0.073)$ g	This is fine. Slightly better would be to round off one more place.
6. $(12.14 \pm$ $0.07)$	Almost there, but what pray tell are the units?
7. (12.1 ± 0.1) g	Wrong. You went overboard in rounding. Stop when the uncertainty is 0.07, one significant figure.
8. 12.14 g $\pm$ 0.07 g	Almost right. The answer and uncertainty should be in parentheses with unit outside.
9. $(12.14 \pm$ $0.07)$ g	Correct!

Problem: I measure a length with a meter stick with a least count of 1 mm. I measure the length 5 times with results (in mm) of 123, 123, 123, 123, 123. What is the average length and the uncertainty in length?

 Answer

Length, L (mm)	$\left| L - L_{ave} \right|$	$\left(L - L_{ave} \right)^2$
123	0.0	0.0
123	0.0	0.0
123	0.0	0.0

123	0.0	0.0
123	0.0	0.0
Sum 616	Sum 0.0	Sum 0.0
Average 123	Average 0.0	St. Dev. 0.0

Here the average deviation and the standard deviation are smaller than the ILE of 0.5 mm. Hence I use 0.5 mm as the unceratinty.

The object has a length of (123.0 +/- 0.5) mm.

(e) Why make many measurements? Standard Error in the Mean.

We know that by making several measurements (4 or 5) we should be more likely to get a good average value for what we are measuring. Is there any point to measuring a quantity more often than this? When you take a statistics course you will learn that the **standard error in the mean** is affected by the number of measurements made.

The standard error in the mean in the simplest case is defined as the standard deviation divided by the square root of the number of measurements.

The following example illustrates this in its simplest form. I am measuring the length of an object. Notice that the average and standard deviation do not change much as the number of measurements change, but that the standard error does dramatically decrease as N increases.

Finding Standard Error in the Mean			
Number of Measurements, N	Average	Standard Deviation	Standard Error
5	15.52 cm	1.33 cm	0.59 cm
25	15.46 cm	1.28 cm	0.26 cm

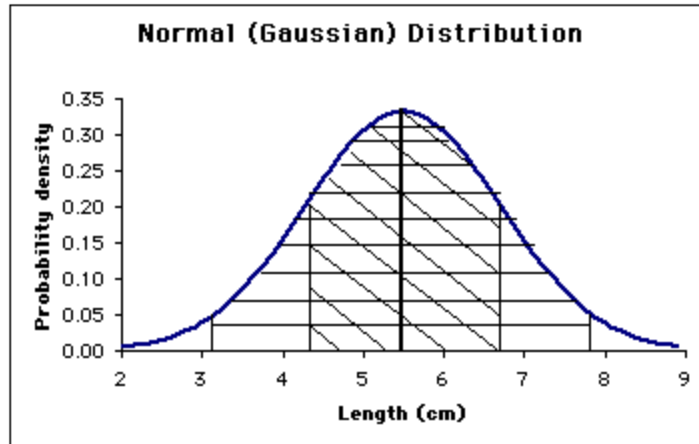| 625 | 15.49 cm | 1.31 cm | 0.05 cm |
| 10000 | 15.49 cm | 1.31 cm | 0.013 cm |

For this introductory course we will not worry about the standard error, but only use the standard deviation, or estimates of the uncertainty.

3. What is the range of possible values?

When you see a number reported as (7.6 ± 0.4) sec your first thought might be that all the readings lie between 7.2 sec (=7.6-0.4) and 8.0 sec (=7.6+0.4). A quick look at the data in the Table 1 shows that this is not the case: only 2 of the 4 readings are in this range. Statistically we expect 68% of the values to lie in the range of $\langle x \rangle \pm \Delta x$, but that 95% lie within $\langle x \rangle \pm 2 \Delta x$. In the first example all the data lie between 6.8 (= 7.6 - 2*0.4) and 8.4 (= 7.6 + 2*0.4) sec. In the second example, 5 of the 6 values lie within two deviations of the average. **As a rule of thumb for this course we usually expect the actual value of a measurement to lie within two deviations of the mean.** If you take a statistics course you will talk about **confidence levels**.

How do we use the uncertainty? Suppose you measure the density of calcite as (2.65 ± 0.04) g / cm^3. The textbook value is 2.71 g / cm^3. Do the two values agree? Since the text value is within the range of two deviations from the average value you measure you claim that your value agrees with the text. If you had measured the density to be (2.65 ± 0.01) g / cm^3 you would be forced to admit your value disagrees with the text value.

The drawing below shows a **Normal Distribution** (also called a **Gaussian**). The vertical axis represents the fraction of measurements that have a given value z. The most likely value is the average, in this case $\langle z \rangle$ = 5.5 cm. The standard deviation is σ = 1.2. The central shaded region is the area under the curve between ($\langle x \rangle$ - σ) and (x + σ), and roughly 67% of the time a measurement will be in this range. The wider shaded region represents ($\langle x \rangle$ - 2σ) and (x + 2σ), and 95% of the measurements will be in this range. A statistics course will go into much more detail about this.

Normal (Gaussian) Distribution

Problem: You measure a time to have a value of (9.22 ± 0.09) s. Your friend measures the time to be (9.385 ± 0.002) s. The accepted value of the time is 9.37 s. Does your time agree with the accepted? Does your friend's time agree with the accepted?

Answer.

We look within 2 deviations of your value, that is between 9.22 - 2(0.09) = 9.04 s and 9.22 + 2(0.09) = 9.40 s. The accepted value is within this range of 9.04 to 9.40 s, so **your experiment agrees with the accepted.**

The news is not so good for your friend. 9.385 - 2(0.002) = 9.381 s and 9.385 + 2(0.002) = 9.389 s. The range of answers for your friend, 9.381 to 9.389 s, does not include the accepted value, so **your friend's time does not agree with the accepted value.**

Problem: Are the following numbers equal within the expected range of values?

(i) (3.42 ± 0.04) m/s and 3.48 m/s?
(ii) (13.106 ± 0.014) grams and 13.206 grams?
(iii) $(2.95 \pm 0.03) \times 10^{8}$ m/s and 3.00×10^{8} m/s

Answer

(i) (3.42 ± 0.04) m/s and 3.48 m/s?

4. Relative and Absolute Errors

The quantity Δz is called the **absolute error** while $\Delta z/z$ is called the **relative error** or **fractional uncertainty**. **Percentage error** is the fractional error multiplied by 100%. In practice, either the percentage error or the absolute error may be provided. Thus in machining an engine part the **tolerance** is usually given as an absolute error, while electronic components are usually given with a percentage tolerance.

5. Propagation of Errors, Basic Rules

Suppose two measured quantities x and y have uncertainties, Δx and Δy, determined by procedures described in previous sections: we would report (x ± Δx), and (y ± Δy). From the measured quantities a new quantity, z, is calculated from x and y. What is the uncertainty, Δz, in z? For the purposes of this course we will use a simplified version of the proper statistical treatment. The formulas for a full statistical treatment (using standard deviations) will also be

given. The guiding principle in all cases is to consider the most pessimistic situation. Full explanations are covered in statistics courses.

The examples included in this section also show the proper rounding of answers, which is covered in more detail in Section 6. The examples use the propagation of errors using average deviations.

(a) Addition and Subtraction: z = x + y   or   z = x - y

Derivation: We will assume that the uncertainties are arranged so as to make z as far from its true value as possible.

Average deviations $\Delta z = |\Delta x| + |\Delta y|$ in both cases

With more than two numbers added or subtracted we continue to add the uncertainties.

Using simpler average errors		Using standard deviations	
$\Delta z \;=\; |\Delta x| \;+\; |\Delta y| \;+\; ...$	Eq. 1a	$\Delta z = \sqrt{(\Delta x)^2 + (\Delta y)^2 + ...}$	Eq. 1b

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Example: w = (4.52 ± 0.02) cm, x = (2.0 ± 0.2) cm, y = (3.0 ± 0.6) cm. Find z = x + y - w and its uncertainty.

z = x + y - w = 2.0 + 3.0 - 4.5 = 0.5 cm

$\Delta z = \Delta x + \Delta y + \Delta w = 0.2 + 0.6 + 0.02 =$ 0.82 rounding to 0.8 cm So z = (0.5 ± 0.8) cm	Solution with standard deviations, Eq. 1b, $\Delta z = 0.633$ cm z = (0.5 ± 0.6) cm

Notice that we round the uncertainty to one significant figure and round the answer to match.

For **multiplication by an exact number**, multiply the uncertainty by the same exact number.

Example: The radius of a circle is x = (3.0 ± 0.2) cm. Find the circumference and its uncertainty.

C = 2 π x = 18.850 cm

ΔC = 2 π Δx = 1.257 cm (The factors of 2 and π are exact)

C = (18.8 ± 1.3) cm

We round the uncertainty to two figures since it starts with a 1, and round the answer to match.

Example: x = (2.0 ± 0.2) cm, y = (3.0 ± 0.6) cm. Find z = x - 2y and its uncertainty.

z = x - 2y = 2.0 - 2(3.0) = -4.0 cm Δz = Δx + 2 Δy = 0.2 + 1.2 = 1.4 cm So z = (-4.0 ± 1.4) cm.	Using Eq 1b, z = (-4.0 ± 0.9) cm.

The 0 after the decimal point in 4.0 is significant and must be written in the answer. The uncertainty in this case starts with a 1 and is kept to two significant figures. (More on rounding in Section 7.)

(b) Multiplication and Division: z = x y   or   z = x/y

Derivation: We can derive the relation for multiplication easily. Take the largest values for x and y, that is

$z + \Delta z = (x + \Delta x)(y + \Delta y) = xy + x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y$

Usually $\Delta x \ll x$ and $\Delta y \ll y$ so that the last term is much smaller than the other terms and can be neglected. Since $z = xy$,

$\Delta z = y\,\Delta x + x\,\Delta y$

which we write more compactly by forming the relative error, that is the ratio of $\Delta z/z$, namely

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \dots$$

The same rule holds for multiplication, division, or combinations, namely add all the relative errors to get the relative error in the result.

Using simpler average errors		Using standard deviations	
$\dfrac{\Delta z}{z} = \dfrac{\Delta x}{x} + \dfrac{\Delta y}{y} + \dots$	Eq. 2a	$\dfrac{\Delta z}{z} = \sqrt{\left(\dfrac{\Delta x}{x}\right)^2 + \left(\dfrac{\Delta y}{y}\right)^2 + \dots}$	Eq.2b

Example: $w = (4.52 \pm 0.02)$ cm, $x = (2.0 \pm 0.2)$ cm. Find $z = w\,x$ and its uncertainty.

$z = w\,x = (4.52)\,(2.0) = 9.04\ cm^2$

$\dfrac{\Delta z}{9.04\ cm^2} = \dfrac{0.02\ cm}{4.52\ cm} + \dfrac{0.2\ cm}{2.0\ cm} = 0.1044$	Using Eq. 2b we get
So $\Delta z = 0.1044\,(9.04\ cm^2) = 0.944$ which we	

| round to 0.9 cm^2,

$z = (9.0 \pm 0.9)$ cm^2. | $\Delta z =$
0.905
cm^2
and
$z = (9.0 \pm 0.9)$
cm^2. |

The uncertainty is rounded to one significant figure and the result is rounded to match. We write 9.0 cm^2 rather than 9 cm^2 since the 0 is significant.

Example: x = (2.0 ± 0.2) cm, y = (3.0 ± 0.6) sec Find z = x/y.

| z = 2.0/3.0 = 0.6667 cm/s.

So Δz = 0.3 (0.6667 cm/sec)
= 0.2 cm/sec
z = (0.7 ± 0.2) cm/sec | Using Eq. 2b we get z = (0.67 ± 0.15) cm/sec |

Note that in this case we round off our answer to have no more decimal places than our uncertainty.

(c) Products of powers: $z = x^m \, y^n$.

The results in this case are

Using simpler average errors		Using standard deviations	
$\dfrac{\Delta z}{z} = \left\lvert m \right\rvert \dfrac{\Delta x}{x} + \left\lvert n \right\rvert \dfrac{\Delta y}{y} + \dots$	Eq. 3a	$\dfrac{\Delta z}{z} = \sqrt{\left(\dfrac{m\,\Delta x}{x}\right)^2 + \left(\dfrac{n\,\Delta y}{y}\right)^2 + \dots}$	Eq.3b

Example: $w = (4.52 \pm 0.02)$ cm, $A = (2.0 \pm 0.2)\ cm^2$, $y = (3.0 \pm 0.6)$ cm.

Find $z = w\, y^2 / \sqrt{A}$.

$$z = \frac{wy^2}{\sqrt{A}} = \frac{4.5cm\ (3.02cm)^2}{\sqrt{2.0\ cm^2}} = 28.638\ cm^2$$

$$\frac{\Delta z}{28.638\ cm^2} = \frac{0.2\ cm}{4.5\ cm} + 2\frac{0.6\ cm}{3.0\ cm} + 0.5\frac{0.2\ cm^2}{2.0\ cm^2} = 0.49$$

The second relative error, ($\Delta y/y$), is multiplied by 2 because the power of y is 2.

The third relative error, ($\Delta A/A$), is multiplied by 0.5 since a square root is a power of one half.

So $\Delta z = 0.49$ ($28.638\ cm^2$) $= 14.03\ cm^2$ which we round to $14\ cm^2$

$z = (29 \pm 14)\ cm^2$

Using Eq. 3b, z=(29 ± 12) cm^2

Because the uncertainty begins with a 1, we keep two significant figures and round the answer to match.

(d) Mixtures of multiplication, division, addition, subtraction, and powers.

If z is a function which involves several terms added or subtracted we must apply the above rules carefully. This is best explained by means of an example.

Example: $w = (4.52 \pm 0.02)$ cm, $x = (2.0 \pm 0.2)$ cm, $y = (3.0 \pm 0.6)$ cm. Find $z = w\,x + y^2$

$z = wx + y^{\wedge 2} = 18.0\ cm^2$

First we compute v = wx as in the example in (b) to get v = (9.0 ± 0.9) cm^2.

Next we compute
$$\frac{\Delta(y^2)}{y^2} = \frac{2\Delta y}{y} = \frac{2(0.6\ cm)}{3.0\ cm} = 0.40$$
$$\Delta(y^2) = 0.40(9.00\ cm^2) = 3.6\ cm^2$$

We have v = wx = (9.0 ± 0.9) cm.

The calculation of the uncertainty in y^2 is the same as that shown to the left. Then from Eq. 1b

Finally, we compute $\Delta z = \Delta v + \Delta(y^2) = 0.9 + 3.6 = 4.5 \ cm^2$ rounding to 4 cm^2 Hence $z = (18 \pm 4) \ cm^2$.	$\Delta z = 3.7 \ cm^2$ $z = (18 \pm 4) \ cm^2$.

(e) Other Functions: e.g.. z = sin x. The simple approach.

For other functions of our variables such as sin(x) we will not give formulae. However you can estimate the error in z = sin(x) as being the difference between the largest possible value and the average value. and use similar techniques for other functions.
Thus

$\Delta(\sin x) = \sin(x + \Delta x) - \sin(x)$

Example: Consider $S = x \cos(\theta)$ for $x = (2.0 \pm 0.2)$ cm, $\theta = 53 \pm 2$ °. Find S and its uncertainty.

$S = (2.0 \text{ cm}) \cos 53° = 1.204$ cm

To get the largest possible value of S we would make x larger, $(x + \Delta x) = 2.2$ cm, and θ smaller, $(\theta - \Delta \theta) = 51°$. The largest value of S, namely $(S + \Delta S)$, is $(S + \Delta S) = (2.2 \text{ cm}) \cos 51° = 1.385$ cm.

The difference between these numbers is $\Delta S = 1.385 - 1.204 = 0.181$ cm which we round to 0.18 cm.

Then $S = (1.20 \pm 0.18)$ cm.

** (f) Other Functions: Getting formulas using partial derivatives

The general method of getting formulas for propagating errors involves the total differential of a function. Suppose that z = f(w, x, y, ...) where the variables w, x, y, etc. must be **independent variables**!

The total differential is then

$$dz = \left(\frac{\partial f}{\partial w}\right) dw + \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \ldots$$

We treat the $dw = \Delta w$ as the error in w, and likewise for the other differentials, dz, dx, dy, etc. The numerical values of the partial derivatives are evaluated by using the average values of w, x,

y, etc. The general results are

Using simpler average errors	
$\Delta z = \left\|\dfrac{\partial f}{\partial w}\right\| \Delta w + \left\|\dfrac{\partial f}{\partial x}\right\| \Delta x + \left\|\dfrac{\partial f}{\partial y}\right\| \Delta y + \ldots$	Eq. 4a.
Using standard deviations	
$\Delta z^2 = \left(\dfrac{\partial f}{\partial w}\right)^2 \Delta w^2 + \left(\dfrac{\partial f}{\partial x}\right)^2 \Delta x^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 \Delta y^2 + \ldots$	Eq. 4b

Example: Consider $S = x \cos(\theta)$ for $x = (2.0 \pm 0.2)$ cm, $\theta = (53 \pm 2)$ °$=$ (0.9250 ± 0.0035) rad. Find S and its uncertainty. Note: **the uncertainty in angle must be in radians!**

$S = 2.0$ cm cos 53° $= 1.204$ cm

$dS = \cos\theta \; dx - x \sin\theta \; d\theta$

We take the absolute value of each term.

$\Delta S = \cos\theta \; \Delta x + x \sin\theta \; \Delta\theta$

$\quad = \cos 53°\,(0.2 \; cm) + (2 \; cm)(\sin 53°)(0.0035) = 0.126 \; cm$

For standard deviation approach,

$\Delta S = \sqrt{(\cos\theta\Delta x)^2 + (-x\sin\theta\Delta\theta)^2}$

$\quad = 0.120 cm$

Hence $S = (1.20 \pm 0.13)$ cm (using average deviation approach) or $S = (1.20 \pm 0.12)$ cm (using standard deviation approach.)

6. Rounding off answers in regular and scientific notation.

In the above examples we were careful to round the answers to an appropriate number of significant figures. The uncertainty should be rounded off to one or two significant figures. If the leading figure in the uncertainty is a 1, we use two significant figures, otherwise we use one significant figure. Then the answer should be rounded to match.

Example Round off z = 12.0349 cm and Δz = 0.153 cm.

Since Δz begins with a 1, we round off Δz to two significant figures:

Δz = 0.15 cm. Hence, round z to have the same number of decimal places:

z = (12.03 ± 0.15) cm.

When the answer is given in scientific notation, the uncertainty should be given in scientific notation with the **same power of ten**. Thus, if

$z = 1.43 \times 10^6$ s and $\Delta z = 2 \times 10^4$ s,

we should write our answer as

$z = (1.43 \pm 0.02) \times 10^6$ s.

This notation makes the range of values most easily understood. The following is technically correct, but is hard to understand at a glance.

$z = (1.43 \times 10^6 \pm 2 \times 10^4)$ s. ***Don't write like this!***

Problem: Express the following results in proper rounded form, x ± Δx.

(i) m = 14.34506 grams, Δm = 0.04251 grams.
(ii) t = 0.02346 sec, Δt = 1.623 x 10^{-3}sec.
(iii) M = 7.35 x 10^{22} kg ΔM = 2.6 x 10^{20} kg.
(iv) m = 9.11 x 10^{-33} kg Δm = 2.2345 x 10^{-33} kg

(i) m = 14.34506 grams, Δm = 0.04251 grams.	m = (14.35 +/- 0.04) g
(ii) t = 0.02346 sec, Δt = 1.623 x 10⁻³ sec.	t = (0.0235 +/- 0.0016) s **or** t = (2.35 +/- 0.16) x 10⁻³s **or** (2.35 +/- 0.16) ms
(iii) M = 7.35 x 10²² kg ΔM = 2.6 x 10²⁰ kg.	M = (7.35 +/- 0.03) x 10²² kg
(iv) m = 9.11 x 10⁻³³ kg Δm = 2.2345 x 10⁻³³ kg	m = (9 +/- 2) x 10⁻³³kg

7. Significant Figures

The rules for propagation of errors hold true for cases when we are in the lab, but doing propagation of errors is time consuming. The rules for **significant figures** allow a much quicker method to get results that are approximately correct even when we have no uncertainty values.

A significant figure is any digit 1 to 9 and any zero which is not a place holder. Thus, in 1.350 there are 4 significant figures since the zero is not needed to make sense of the number. In a number like 0.00320 there are 3 significant figures --the first three zeros are just place holders. However the number 1350 is ambiguous. You cannot tell if there are 3 significant figures --the 0 is only used to hold the units place --or if there are 4 significant figures and the zero in the units place was actually measured to be zero.

How do we resolve ambiguities that arise with zeros when we need to use zero as a place holder as well as a significant figure? Suppose we measure a length to three significant figures as 8000 cm. Written this way we cannot tell if there are 1, 2, 3, or 4 significant figures. To make the number of significant figures apparent we use scientific notation, 8×10^3 cm (which has one significant figure), or 8.00×10^3 cm (which has three significant figures), or whatever is correct under the circumstances.

We start then with numbers each with their own number of significant figures and compute a new quantity. How many significant figures should be in the final answer? In doing running computations we maintain numbers to many figures, but we must report the answer only to the proper number of significant figures.

In the case of addition and subtraction we can best explain with an example. Suppose one object is measured to have a mass of 9.9 gm and a second object is measured on a different balance to have a mass of 0.3163 gm. What is the total mass? We write the numbers with question marks at places where we lack information. Thus 9.9???? gm and 0.3163? gm. Adding them with the decimal points lined up we see

```
09.9????
00.3163?
10.2???? = 10.2 gm.
```

In the case of multiplication or division we can use the same idea of unknown digits. Thus the product of 3.413? and 2.3? can be written in long hand as

```
3.413?
2.3?
   ?????
 10239?
 6826?
7.8????? = 7.8
```

The short rule for multiplication and division is that the answer will contain a number of significant figures equal to the number of significant figures in the entering number having the least number of significant figures. In the above example 2.3 had 2 significant figures while 3.413 had 4, so the answer is given to 2 significant figures.

It is important to keep these concepts in mind as you use calculators with 8 or 10 digit displays if you are to avoid mistakes in your answers and to avoid the wrath of physics instructors everywhere. A good procedure to use is to use use all digits (significant or not) throughout calculations, and only round off the answers to appropriate "sig fig."

Problem: How many significant figures are there in each of the following?

(i) 0.00042   (ii) 0.14700   (ii) 4.2 x 10^6 (iv) -154.090 x 10^{-27}

Answer

Question	Number of Significant Figures
(i) 0.00042	2
(ii) 0.14700	5
(ii) 4.2 x 10^6	2
(iv) -154.090 x 10^{-27}	6

8. Problems on Uncertainties and Error Propagation.

Try the following problems to see if you understand the details of this part . The answers are at the end.

(a) Find the average and the average deviation of the following measurements of a mass.

4.32, 4.35, 4.31, 4.36, 4.37, 4.34 grams.

(b) Express the following results in proper rounded form, $x \pm \Delta x$.

(i) m = 14.34506 grams, Δm = 0.04251 grams.
(ii) t = 0.02346 sec, Δt = 1.623 x 10^{-3} sec.
(iii) M = 7.35 x 10^{22} kg ΔM = 2.6 x 10^{20} kg.
(iv) m = 9.11 x 10^{-33} kg Δm = 2.2345 x 10^{-33} kg

(c) Are the following numbers equal within the expected range of values?

(i) (3.42 ± 0.04) m/s and 3.48 m/s?
(ii) (13.106 ± 0.014) grams and 13.206 grams?
(iii) (2.95 ± 0.03) x 10^8 m/s and 3.00 x 10^8 m/s

(d) Calculate z and Δz for each of the following cases.

(i) z = (x - 2.5 y + w) for x = (4.72 ± 0.12) m, y = (4.4 ± 0.2) m, w = (15.63 ± 0.16) m.
(ii) z = (w x/y) for w = (14.42 ± 0.03) m/s^2, x = (3.61 ± 0.18) m, y = (650 ± 20) m/s.
(iii) z = x^3 for x = (3.55 ± 0.15) m.
(iv) z = v (xy + w) with v = (0.644 ± 0.004) m, x = (3.42 ± 0.06) m, y = (5.00 ± 0.12) m, w
= (12.13 ± 0.08) m^2.
(v) z = A sin y for A = (1.602 ± 0.007) m/s, y = (0.774 ± 0.003) rad.

(e) How many significant figures are there in each of the following?

(i) 0.00042   (ii) 0.14700   (ii) 4.2 x 10^6 (iv) -154.090 x 10^{-27}

(f) I measure a length with a meter stick which has a least count of 1 mm I measure the length 5 times with results in mm of 123, 123, 124, 123, 123 mm. What is the average length and the uncertainty in length?

(a) (4.342 ± 0.018) grams

(b) i) (14.34 ± 0.04) grams   ii) (0.0235 ± 0.0016) sec or $(2.35 \pm 0.16) \times 10^{-2}$ sec

   iii) $(7.35 \pm 0.03) \times 10^{22}$ kg   iv) $(9.11 \pm 0.02) \times 10^{-31}$ kg

(c) Yes for (i) and (iii), no for (ii)

(d) i) (9.4 ± 0.8) m   ii) (0.080 ± 0.007) m/s   iii) (45 ± 6) m^3 iv) $18.8 \pm 0.6)$ m^3 v) $(1.120 \pm 0.008$ m/s

(e) i) 2   ii) 5   iii) 2   iv) 6

(f) (123 ± 1) mm (I used the ILE = least count since it is larger than the average deviation.)

9. Glossary of Important Terms

Term	Brief Definition
Absolute error	The actual error in a quantity, having the same units as the quantity. Thus if $c = (2.95 \pm 0.07)$ m/s, the absolute error is 0.07 m/s. See *Relative Error.*
Accuracy	How close a measurement is to being correct. For gravitational acceleration near the earth, $g = 9.7$ m/s^2 is more accurate than $g = 9.532706$ m/s^2. See *Precision*.
Average	When several measurements of a quantity are made, the sum of the measurements divided by the number of measurements.
Average Deviation	The average of the absolute value of the differences between each measurement and the average. See *Standard Deviation*.
Confidence Level	The fraction of measurements that can be expected to lie within a given range. Thus if m = (15.34 ± 0.18) g, at 67% confidence level, 67% of the measurements lie within (15.34 - 0.18) g and (15.34 + 0.18) g. If we use 2 deviations (±0.36 here) we have a 95% confidence level.
Deviation	A measure of range of measurements from the average. Also called *error oruncertainty*.
Error	A measure of range of measurements from the average. Also called *deviation* or *uncertainty*.
Estimated Uncertainty	An uncertainty estimated by the observer based on his or her knowledge of the experiment and the equipment. This is in contrast to *ILE*, *standard deviation* or *average deviation*.
Gaussian Distribution	The familiar bell-shaped distribution. Simple statistics assumes that random errors are distributed in this distribution. Also called *Normal Distribution*.
Independent	Changing the value of one variable has no effect on any of the

Variables	other variables. *Propagation of errors* assumes that all variables are independent.
Instrument Limit of Error (ILE)	The smallest reading that an observer can make from an instrument. This is generally smaller than the *Least Count*.
Least Count	The size of the smallest division on a scale. Typically the *ILE* equals the least count or 1/2 or 1/5 of the least count.
Normal Distribution	The familiar bell-shaped distribution. Simple statistics assumes that random errors are distributed in this distribution. Also called *Gaussian Distribution*.
Precision	The number of significant figures in a measurement. For gravitational acceleration near the earth, $g = 9.532706$ m/s^2 is more precise than $g = 9.7$ m/s^2. Greater precision does not mean greater accuracy! See *Accuracy*.
Propagation of Errors	Given independent variables each with an uncertainty, the method of determining an uncertainty in a function of these variables.
Random Error	Deviations from the "true value" can be equally likely to be higher or lower than the true value. See *Systematic Error*.
Range of Possible True Values	Measurements give an average value, <x> and an uncertainty, Δx. At the 67% confidence level the range of possible true values is from <x> - Δx to <x> + Δx. See C*onfidence Level* .
Relative Error	The ratio of absolute error to the average, Δx/x. This may also be called percentage error or fractional uncertainty. See *Absolute Error*.
Significant Figures	All non-zero digits plus zeros that do not just hold a place before or after a decimal point.
Standard Deviation	The statistical measure of uncertainty. See *Average Deviation*.
Standard Error in the Mean	An advanced statistical measure of the effect of large numbers of measurements on the range of values expected for the average (or

	mean).
Systematic Error	A situation where all measurements fall above or below the "true value". Recognizing and correcting systematic errors is very difficult.
Uncertainty	A measure of range of measurements from the average. Also called *deviation* or *error*.