

How to Propagate Uncertainty in Basic Calculations

<http://lectureonline.cl.msu.edu/~mmp/labs/error/e2.htm>

Very often we are facing the situation that we need to measure two or more quantities, each with their individual uncertainties, and then combine the information from these quantities in order to come up with a final result of our experiment. How can you state your answer for the combined result of these measurements and their uncertainties scientifically?

The answer to this fairly common question depends on how the individual measurements are combined in the result. We will treat each case separately:

1. *Addition of measured quantities*

If you have measured values for the quantities X, Y, and Z, with uncertainties δX , δY , and δZ , and your final result, R, is the sum or difference of these quantities, then the uncertainty δR is:

$$\begin{aligned}R &= X + Y - Z \\ \delta R &\approx \delta X + \delta Y + \delta Z \\ \delta R &= \sqrt{(\delta X)^2 + (\delta Y)^2 + (\delta Z)^2}\end{aligned}$$

Here the upper equation is an approximation that can also serve as an upper bound for the error. Please note that the rule is the same for addition and subtraction of quantities.

Example:

Suppose we have measured the starting position as $x_1 = 9.3 \pm 0.2$ m and the finishing position as $x_2 = 14.4 \pm 0.3$ m. Then the displacement is:

$$\Delta x = x_2 - x_1 = 14.4 \text{ m} - 9.3 \text{ m} = 5.1 \text{ m}$$

and the error in the displacement is:

$$(0.2^2 + 0.3^2)^{1/2} \text{ m} = 0.36 \text{ m}$$

2. *Multiplication of measured quantities*

In the same way as for sums and differences, we can also state the result for the case of multiplication and division:

$$R = \frac{X \cdot Y}{Z}$$
$$\frac{\delta R}{|R|} \approx \frac{\delta X}{|X|} + \frac{\delta Y}{|Y|} + \frac{\delta Z}{|Z|}$$
$$\delta R = |R| \cdot \sqrt{\left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2 + \left(\frac{\delta Z}{Z}\right)^2}$$

Again the upper line is an approximation and the lower line is the exact result for independent random uncertainties in the individual variables. And again please note that for the purpose of error calculation there is no difference between multiplication and division.

Example:

We have measured a displacement of $x = 5.1 \pm 0.4$ m during a time of $t = 0.4 \pm 0.1$ s. What is the average velocity and the error in the average velocity?

$$v = x / t = 5.1 \text{ m} / 0.4 \text{ s} = 12.75 \text{ m/s}$$

and the uncertainty in the velocity is:

$$\delta v = |v| \cdot [(\delta x/x)^2 + (\delta t/t)^2]^{1/2} = 12.75 \text{ m/s} \cdot [(0.4/5.1)^2 + (0.1/0.4)^2]^{1/2} = 3.34 \text{ m/s}$$

3. *Multiplication with a constant*

What if you have measured the uncertainty in an observable X , and you need to multiply it with a constant that is known exactly? What is the error then? This is easy: just multiply the error in X with the absolute value of the constant, and this will give you the error in R :

$$R = c \cdot X$$
$$\delta R = |c| \cdot \delta X$$

If you compare this to the above rule for multiplication of two quantities, you see that this is just the special case of that rule for the uncertainty in c , $\delta c = 0$.

Example:

If an object is released from rest and is in free fall, and if you measure the velocity of this object at some point to be $v = -3.8 \pm 0.3$ m/s, how long has it been in free fall?

Answer: we can calculate the time as ($g = 9.81$ m/s² is assumed to be known exactly)

$$t = -v / g = 3.8 \text{ m/s} / 9.81 \text{ m/s}^2 = 0.387 \text{ s}$$

The uncertainty in the fall time is then:

$$\delta t = |-1/g| \cdot \delta v = 0.102 \text{ s}^2/\text{m} \cdot 0.3 \text{ m/s} = 0.03 \text{ s}$$

4. *Polynomial functions*

What if there is a dependence of the result on the measured quantity X that is not described by simple multiplications or additions? We will state the general answer for R as a general function of one or more variables below, but will first cover the special case that R is a polynomial function of one variable X .

$$R = X^n$$
$$\delta R = |n| \cdot \frac{\delta X}{|X|} \cdot |R|$$

You see that this rule is quite simple and holds for positive or negative numbers n , which can even be non-integers.

5. **General functions**

And finally, we can express the uncertainty in R for general functions of one or more observables. If R is a function of X and Y, written as R(X,Y), then the uncertainty in R is obtained by taking the partial derivatives of R with respect to each variable, multiplication with the uncertainty in that variable, and addition of these individual terms in quadrature.

$$R = R(X, Y, \dots)$$
$$\delta R = \sqrt{\left(\frac{\partial R}{\partial X} \cdot \delta X\right)^2 + \left(\frac{\partial R}{\partial Y} \cdot \delta Y\right)^2 + \dots}$$

All rules that we have stated above are actually special cases of this last rule. We leave the proof of this statement as one of those famous "exercises for the reader".