A Fictional Measurement of the Acceleration due to Earth's Gravity

Analysis walk-through, and Excel tutorial

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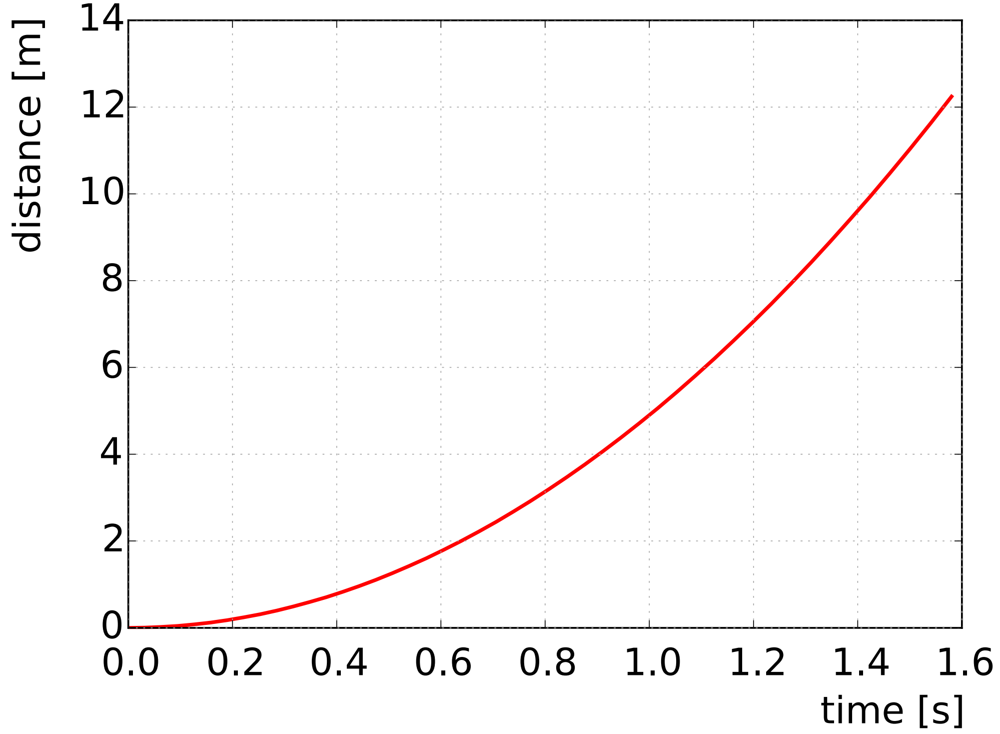
distance is a quadratic function of time!

$$s(t) = \frac{1}{2} g t^2$$

↓

$g = 9.81\ m/s^2$



This is a 1 parameter estimation problem

We have to calculate an estimate of g from our experimental data

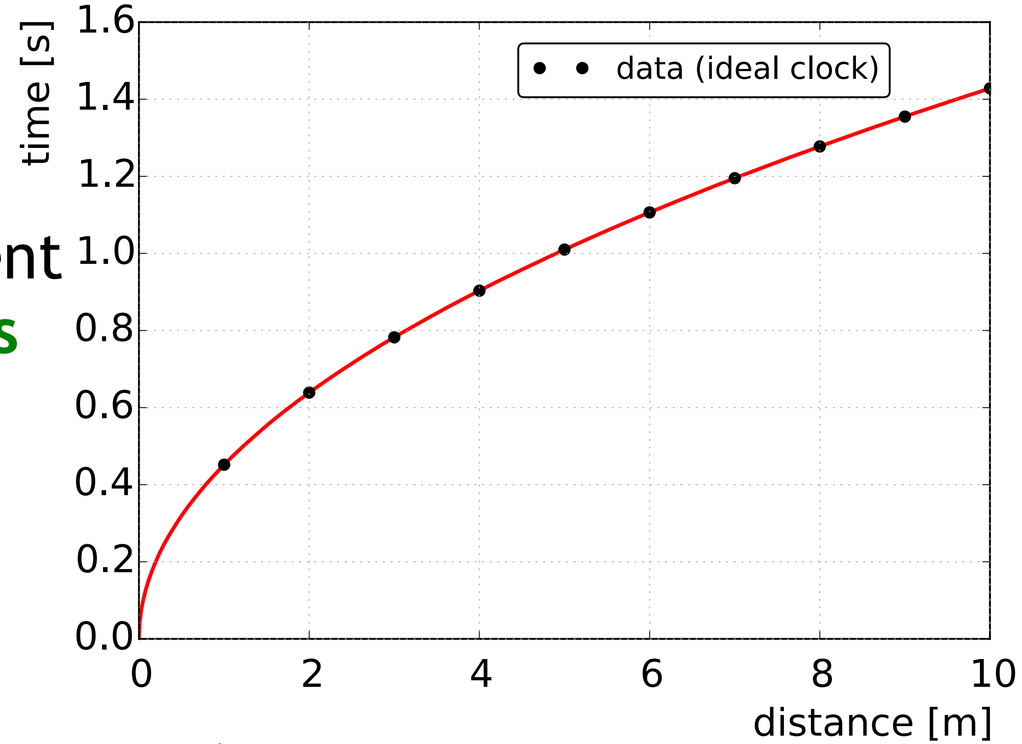
Remember:

In our hypothetical experiment we **actually measure time as function of distance!**

$$s(t) = \frac{1}{2} g t^2$$

Solving for t gives:

$$t(s) = \sqrt{\frac{2s}{g}}$$



g=9.81 m/s²

black points represent our data in **absence of uncertainty** in time measurement (model prediction for data points)

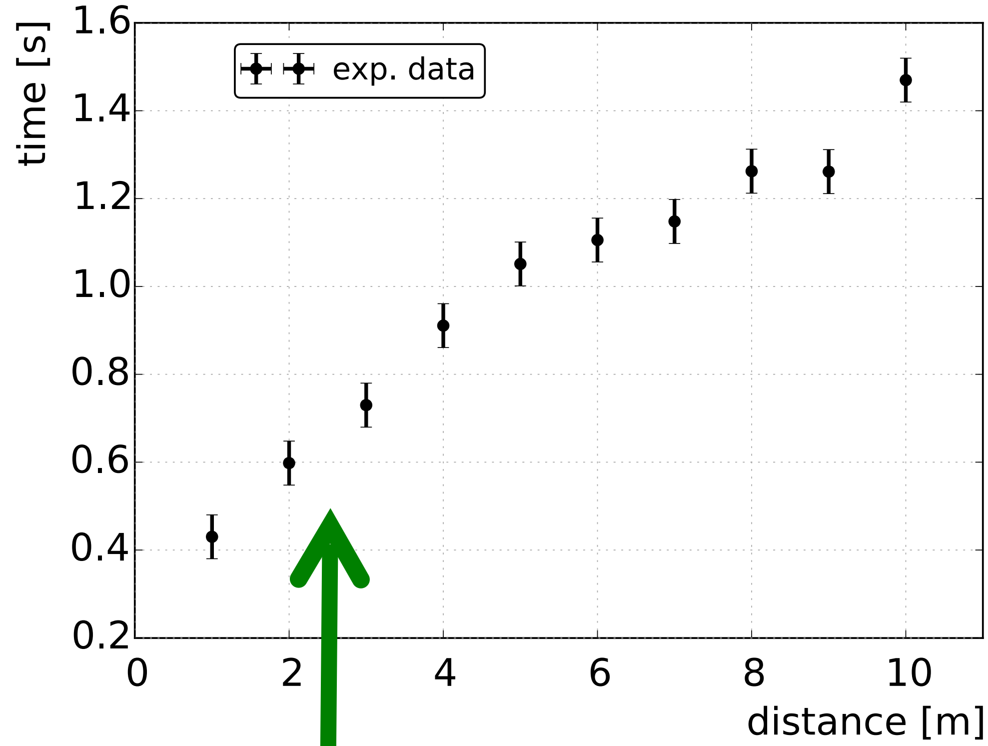
Let’s turn on uncertainty

$$\sigma_t = 0.05\text{s}$$

measured times t_i are now random variables

B11 | f_x | 1.47

	A	B	C	D
1	distance (m)	time (s)		
2	1	0.43		
3	2	0.6		
4	3	0.73		
5	4	0.91		
6	5	1.05		
7	6	1.1		
8	7	1.15		
9	8	1.26		
10	9	1.26		
11	10	1.47		
12				
13				
14				



Actual measurement in experiment with uncertainty

We’ll use the data set shown on the left, in Excel

Need column for uncertainty..

Two methods:
- Create whole column of identical values
- Refer to fixed box, with the one value 0.05

We’ll do the former, for now

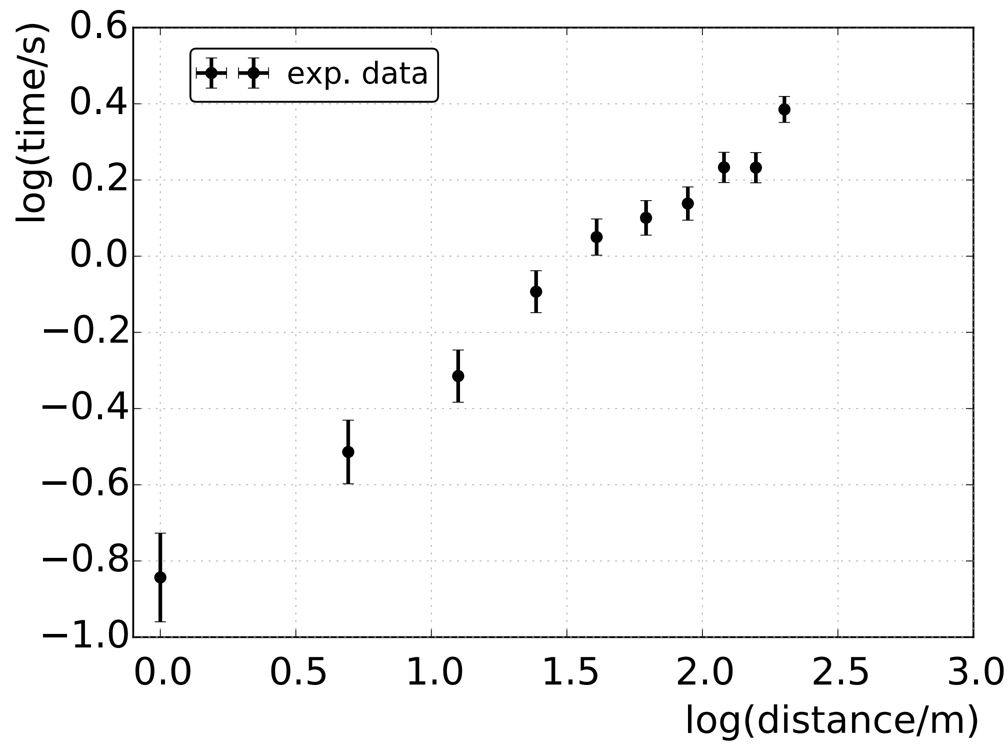
C2 f_x 0.05

	A	B	C	D	E
1	distance (m)	time (s)			
2	1	0.43	0.05		
3	2	0.6			
4	3	0.73			
5	4	0.91			
6	5	1.05			
7	6	1.1			
8	7	1.15			
9	8	1.26			
10	9	1.26			
11	10	1.47			
12				0.05	
13					
14					
15					
16					

you know how to do a "straight line fit". Good!

But time is a non-linear function of distance.

$$t(s) = \sqrt{\frac{2s}{g}}$$



Make the problem linear using logarithms and algebra!

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a^b) = b\ln(a)$$

$$\ln(t) = \frac{1}{2}\ln(s) + \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

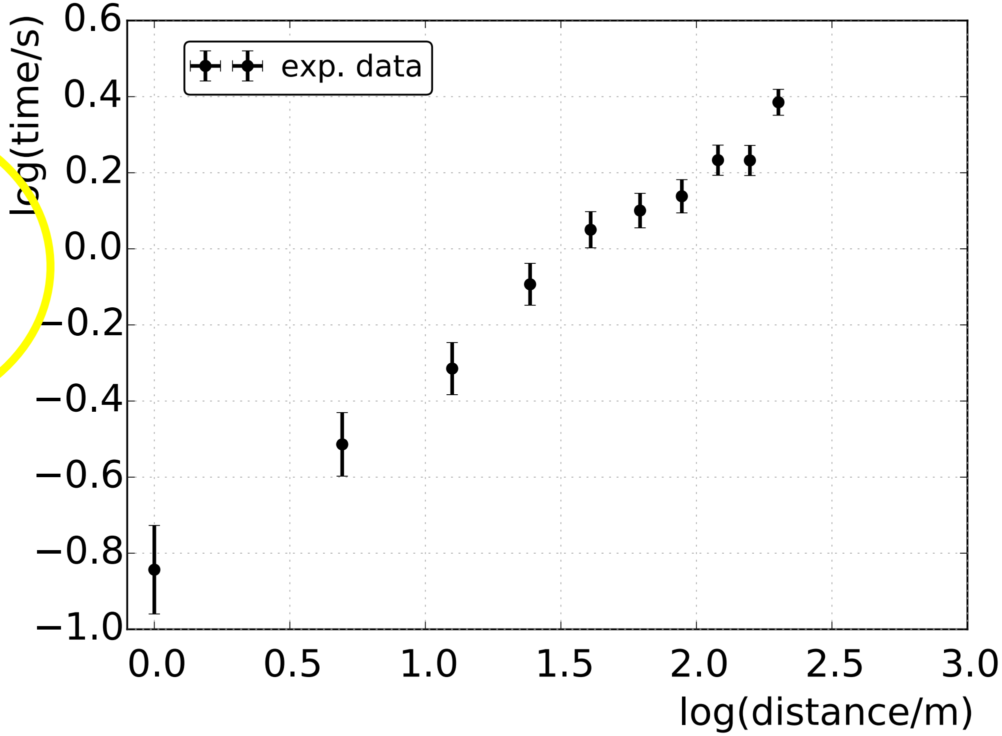
Why can this be viewed as linear?

$$\ln(t) = \frac{1}{2}\ln(s) + \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

compare to

$$y = ax + b$$

and identify



$$x = \ln(s)$$

$$y = \ln(t)$$

$$b = \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

$$g = 2\exp(-2b)$$

Can find g from the offset b of the straight line

Why can this be viewed as linear?

$$\ln(t) = \frac{1}{2}\ln(s) + \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

T.TEST | f_x =LN($A2)

	A	B	C	D	E	F
1	distance (m)	time (s)			log(distance/m)	
2	1	0.43	0.05		=LN($A2)	
3	2	0.6	0.05			
4	3	0.73	0.05			
5	4	0.91	0.05			
6	5	1.05	0.05			
7	6	1.1	0.05			
8	7	1.15	0.05			
9	8	1.26	0.05			
10	9	1.26	0.05			
11	10	1.47	0.05			
12						
13						

compare to

$$y = ax + b$$

and identify

$$x = \ln(s)$$

$$y = \ln(t)$$

So we need to compute new columns for x and y, so that we can do linear fit analysis!

Can find g from the offset b of the straight line

$$\ln(t) = \frac{1}{2}\ln(s) + \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

compare to

$$y = ax + b$$

Now compute:

$$x = \ln(s)$$

$$y = \ln(t)$$

E2 =LN($A2)

	A	B	C	D	E	F
1	distance (m)	time (s)			log(distance/m)	
2	1	0.43	0.05		0	
3	2	0.6	0.05		0.693147	
4	3	0.73	0.05		1.098612	
5	4	0.91	0.05		1.386294	
6	5	1.05	0.05		1.609438	
7	6	1.1	0.05		1.791759	
8	7	1.15	0.05		1.94591	
9	8	1.26	0.05		2.079442	
10	9	1.26	0.05		2.197225	
11	10	1.47	0.05		2.302585	
12						
13						

$$\ln(t) = \frac{1}{2}\ln(s) + \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

compare to

$$y = ax + b$$

Now compute:

$$x = \ln(s)$$

$$y = \ln(t)$$

F2 =LN($B2)

	A	B	C	D	E	F
1	distance (m)	time (s)			log(distance/m)	log(time/s)
2	1	0.43	0.05		0	-0.84397
3	2	0.6	0.05		0.693147181	-0.51083
4	3	0.73	0.05		1.098612289	-0.31471
5	4	0.91	0.05		1.386294361	-0.09431
6	5	1.05	0.05		1.609437912	0.04879
7	6	1.1	0.05		1.791759469	0.09531
8	7	1.15	0.05		1.945910149	0.139762
9	8	1.26	0.05		2.079441542	0.231112
10	9	1.26	0.05		2.197224577	0.231112
11	10	1.47	0.05		2.302585093	0.385262
12						
13						

$$\ln(t) = \frac{1}{2}\ln(s) + \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

compare to

$$y = ax + b$$

Now compute:

$$x = \ln(s)$$

$$y = \ln(t)$$

$$\sigma_s = 0 \rightarrow \sigma_x = 0$$

F2 | f_x | =LN($B2)

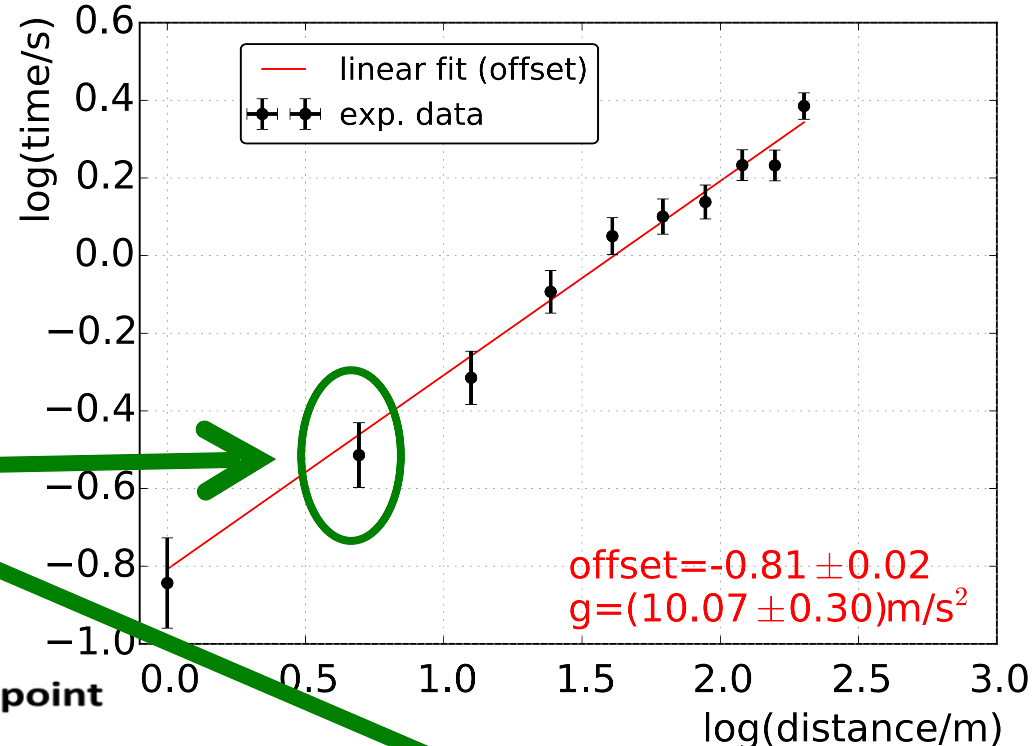
	A	B	C	D	E	F
1	distance (m)	time (s)			log(distance/m)	log(time/s)
2	1	0.43	0.05		0	-0.84397
3	2	0.6	0.05		0.693147181	-0.51083
4	3	0.73	0.05		1.098612289	-0.31471
5	4	0.91	0.05		1.386294361	-0.09431
6	5	1.05	0.05		1.609437912	0.04879
7	6	1.1	0.05		1.791759469	0.09531
8	7	1.15	0.05		1.945910149	0.139762
9	8	1.26	0.05		2.079441542	0.231112
10	9	1.26	0.05		2.197224577	0.231112
11	10	1.47	0.05		2.302585093	0.385262
12						
13						

Need to find uncertainties on $y_i = \ln(t_i)$

from error propagation

$$\sigma_{y_i} = \left|\frac{\partial y_i}{\partial t_i}\right| \sigma_{t_i} = \frac{\sigma_{t_i}}{t_i}$$

Note: the subscript i **refers to each data point**



	A	B	C	D	E	F	G
1	distance (m)	time (s)			log(distance/m)	log(time/s)	sigma_t / t
2	1	0.43	0.05		0	-0.84397007	=$C2/$B2
3	2	0.6	0.05		0.693147181	-0.510825624	
4	3	0.73	0.05		1.098612289	-0.314710745	
5	4	0.91	0.05		1.386294361	-0.094310679	
6	5	1.05	0.05		1.609437912	0.048790164	
7	6	1.1	0.05		1.791759469	0.09531018	
8	7	1.15	0.05		1.945910149	0.139761942	
9	8	1.26	0.05		2.079441542	0.231111721	

Recall:

$$\ln(t) = \frac{1}{2}\ln(s) + \frac{1}{2}\ln\left(\frac{2}{g}\right)$$

$$\Updownarrow$$

$$y = ax + b$$

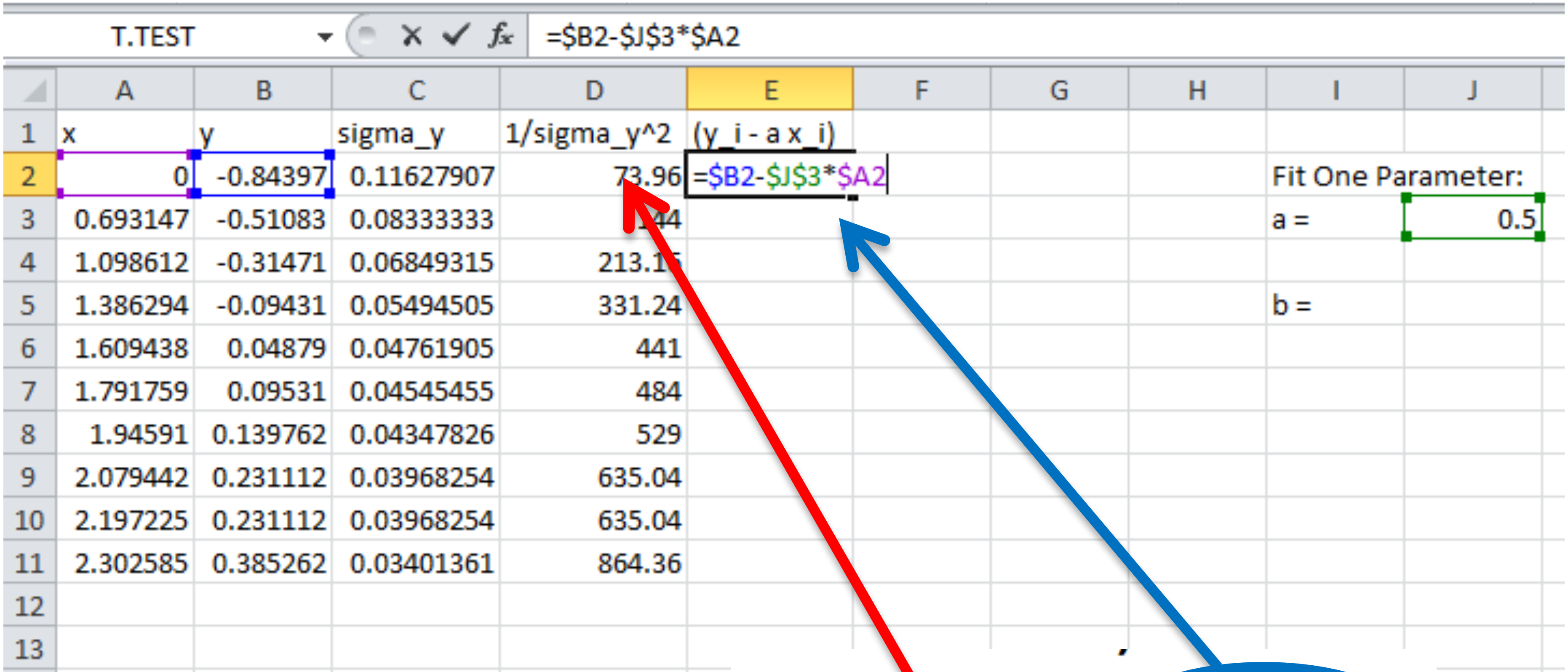
Now we're just doing linear analysis on y, x, with a, and b

But a is fixed to be ½!

This is the case when we only fit one parameter b:

$$b = \frac{1}{\sum \frac{1}{\sigma_{y_i}^2}} \sum \frac{(y_i - ax_i)}{\sigma_{y_i}^2}$$

$$\sigma_b = \sqrt{\frac{1}{\sum \frac{1}{\sigma_{y_i}^2}}}$$

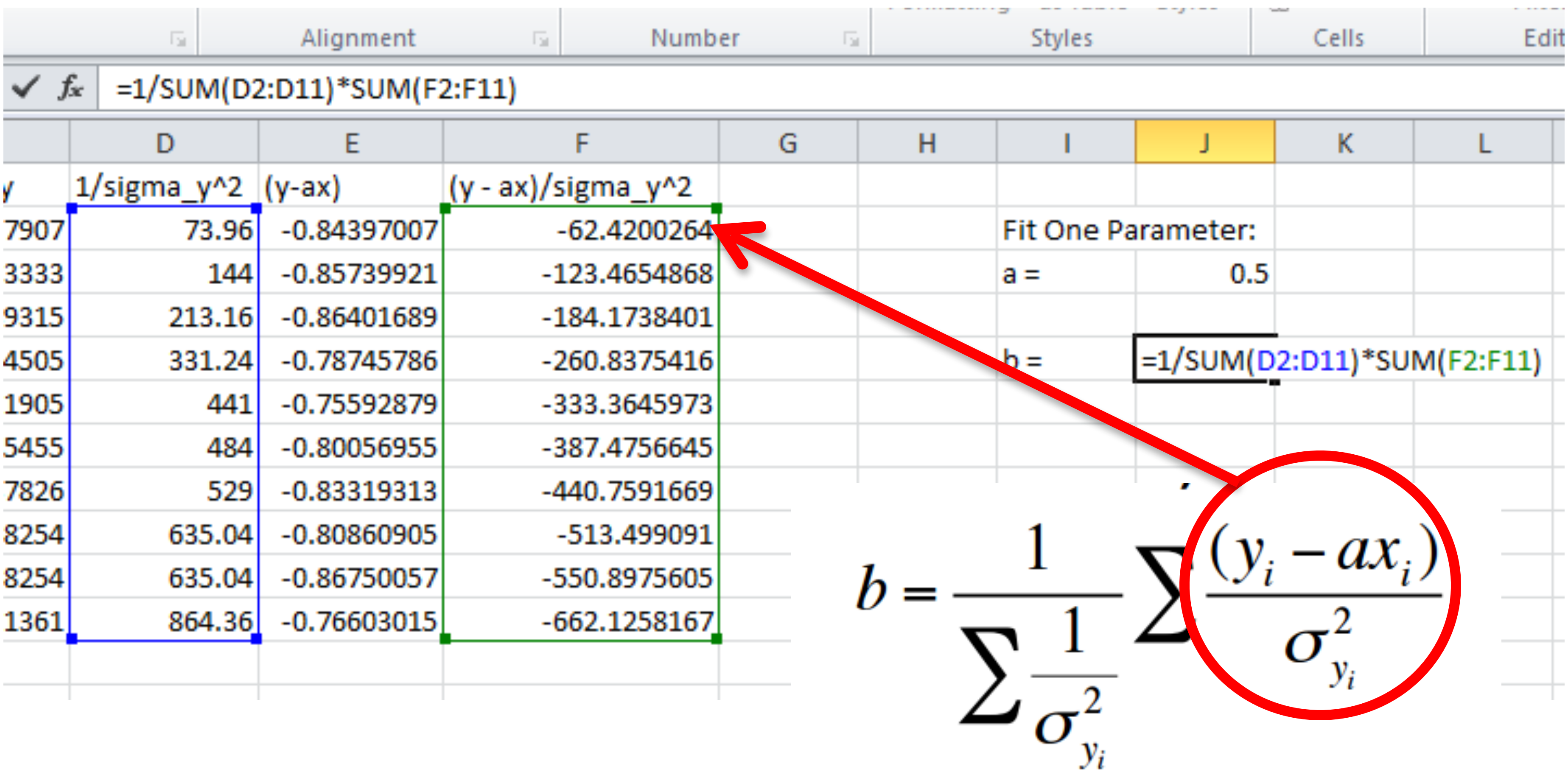


T.TEST | =$B2-$J$3*$A2

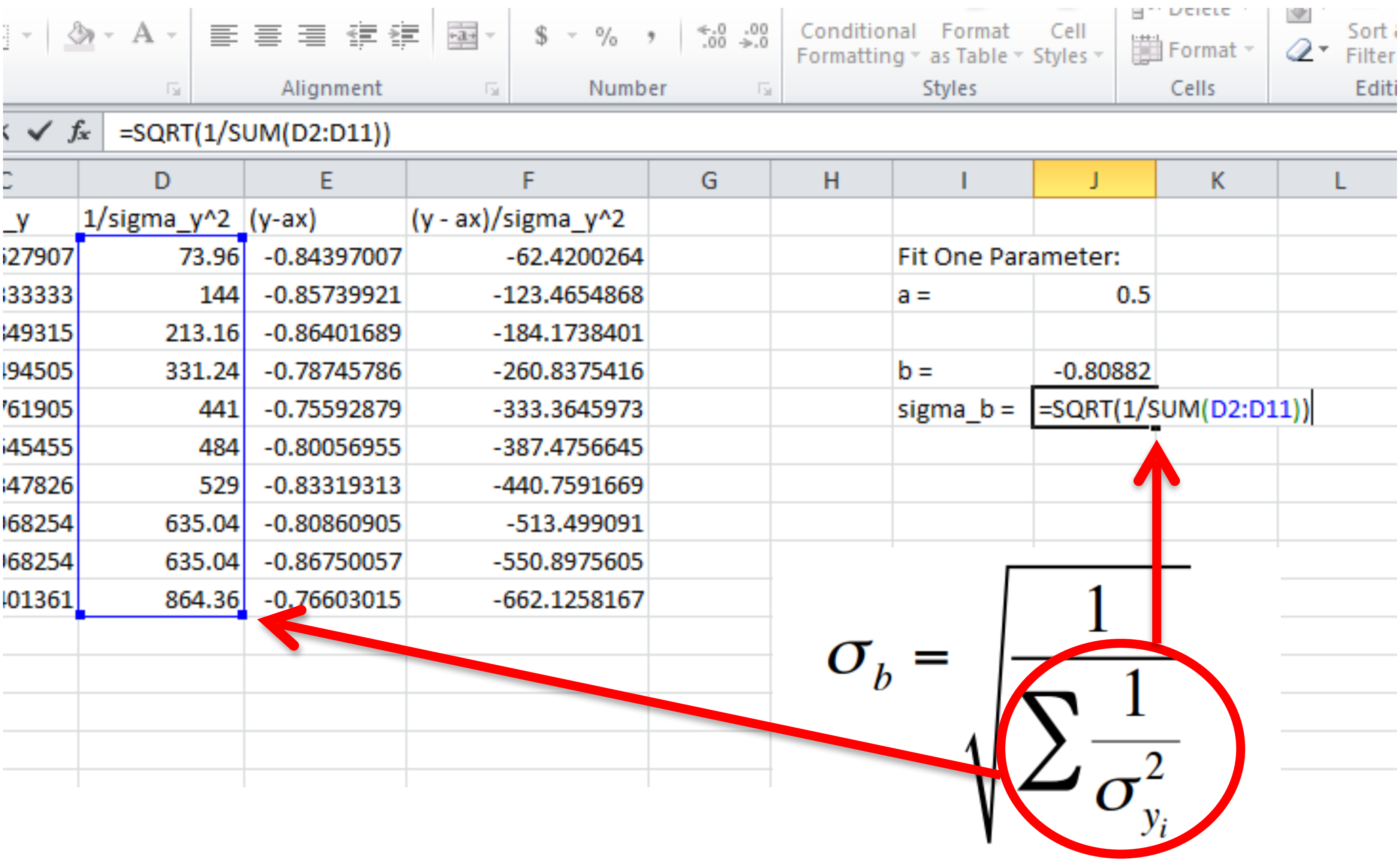
	A	B	C	D	E	F	G	H	I	J
1	x	y	sigma_y	1/sigma_y^2	(y_i - a x_i)					
2	0	-0.84397	0.11627907	73.96	=$B2-$J$3*$A2				Fit One Parameter:	
3	0.693147	-0.51083	0.08333333	144					a =	0.5
4	1.098612	-0.31471	0.06849315	213.16						
5	1.386294	-0.09431	0.05494505	331.24					b =	
6	1.609438	0.04879	0.04761905	441						
7	1.791759	0.09531	0.04545455	484						
8	1.94591	0.139762	0.04347826	529						
9	2.079442	0.231112	0.03968254	635.04						
10	2.197225	0.231112	0.03968254	635.04						
11	2.302585	0.385262	0.03401361	864.36						
12										
13										

Compute this in parts:

$$b = \frac{1}{\sum \frac{1}{\sigma_{y_i}^2}} \sum \frac{(y_i - a x_i)}{\sigma_{y_i}^2}$$



$$b = \frac{1}{\sum \frac{1}{\sigma_{y_i}^2}} \sum \frac{(y_i - ax_i)}{\sigma_{y_i}^2}$$



Conditional Formatting
Format as Table
Cell Styles
Format
Sort & Filter
Alignment
Number
Styles
Cells
=SQRT(1/SUM(D2:D11))
C
D
E
F
G
H
I
J
K
L
1/sigma_y^2
(y-ax)
(y - ax)/sigma_y^2
73.96
-0.84397007
-62.4200264
144
-0.85739921
-123.4654868
213.16
-0.86401689
-184.1738401
331.24
-0.78745786
-260.8375416
441
-0.75592879
-333.3645973
484
-0.80056955
-387.4756645
529
-0.83319313
-440.7591669
635.04
-0.80860905
-513.499091
635.04
-0.86750057
-550.8975605
864.36
-0.76603015
-662.1258167
Fit One Parameter:
a =
0.5
b =
-0.80882
sigma_b =
=SQRT(1/SUM(D2:D11))
$\sigma_b = \sqrt{\dfrac{1}{\sum \dfrac{1}{\sigma_{y_i}^2}}}$

Q: Is this value consistent with $9.81 m/s^2$

Calculate uncertainty on g → easy! (error propagation)

$$\sigma_g = \left|\frac{\partial g}{\partial b}\right| \sigma_b$$

$$\sigma_g = 4\exp(-2b)\sigma_b$$