Physics 252
Guide to measurement and data analysis

Adapted from lectures by Prof. Joanna Kiryluk

Why do we do experiments?

Two types of experiments to learn about the physical world:

R.Muller, "The Instant Physicist"

- parameter determination

  e.g. measure body temperature

- hypothesis testing

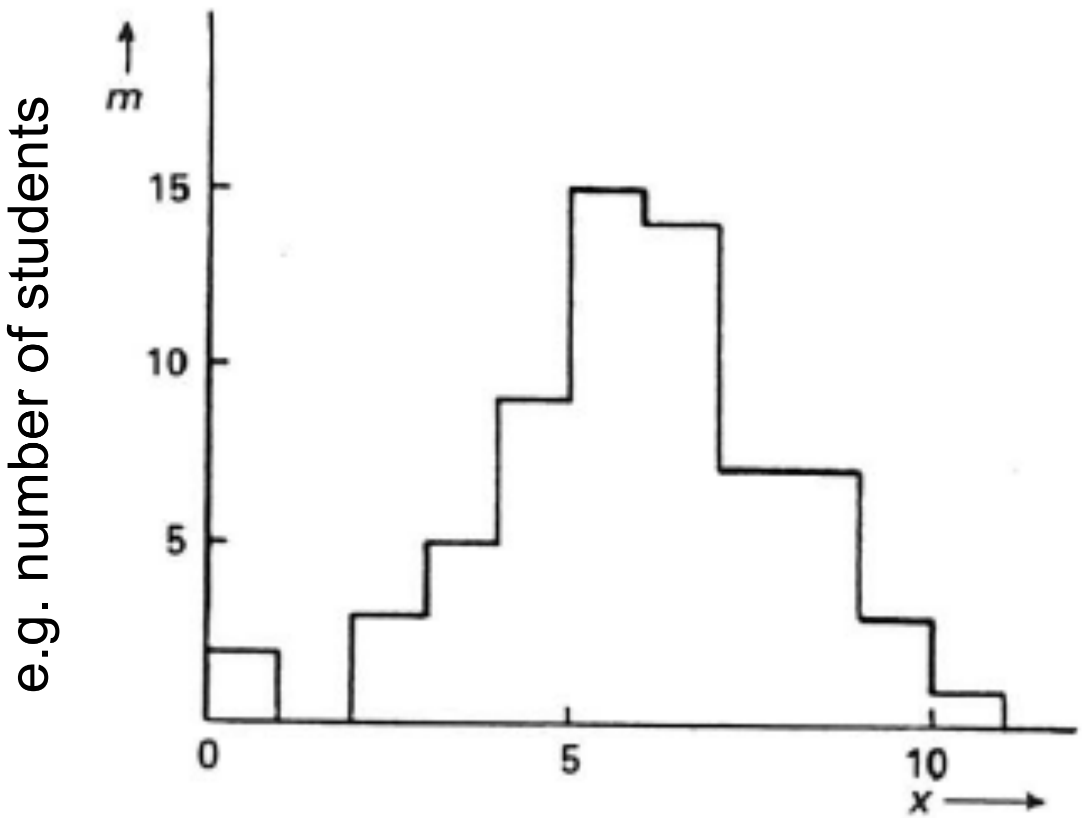
  e.g. testing whether body temperature increased since this morning

The **numerical value** of the quantity we want to measure **is not enough**

Our conclusion, e.g."We have made a world shattering discovery!" depends on the accuracy of our measurement.

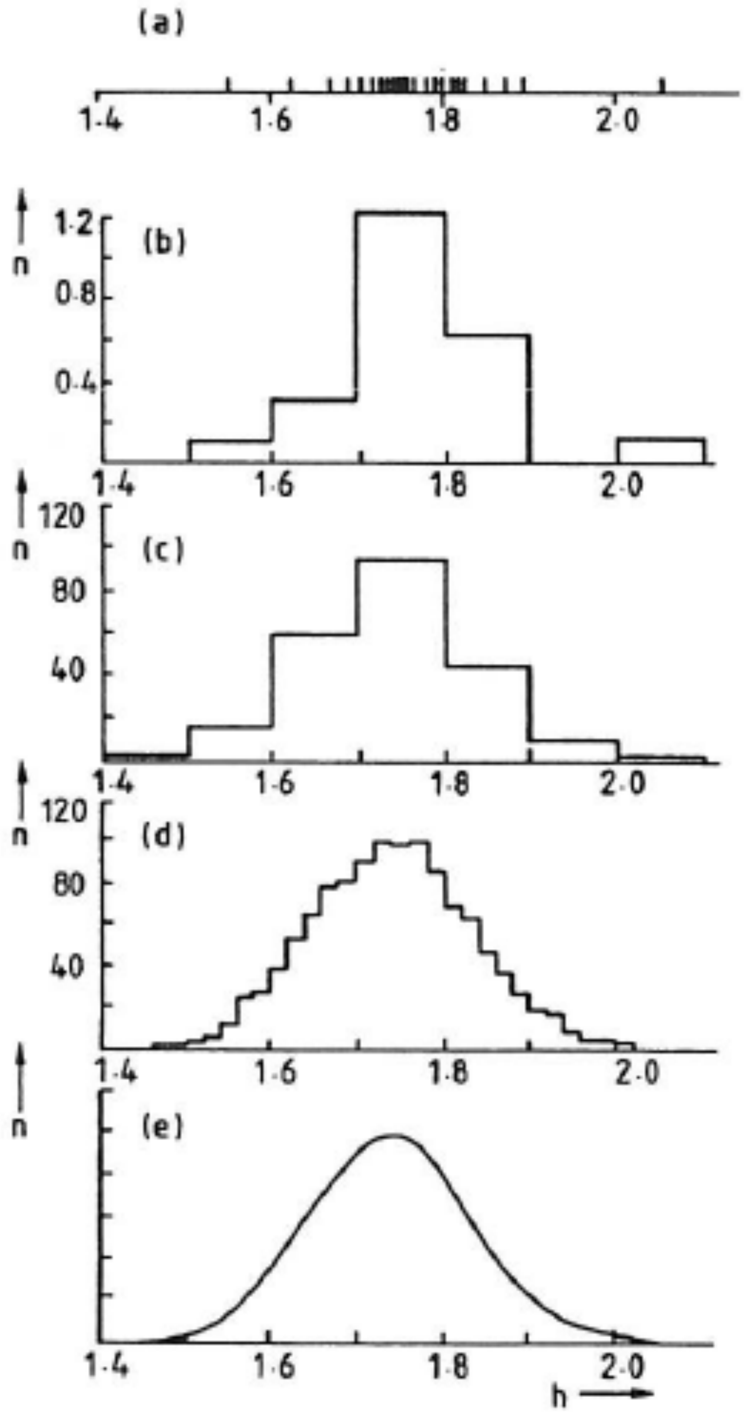
Experimental Data / Results

Histograms

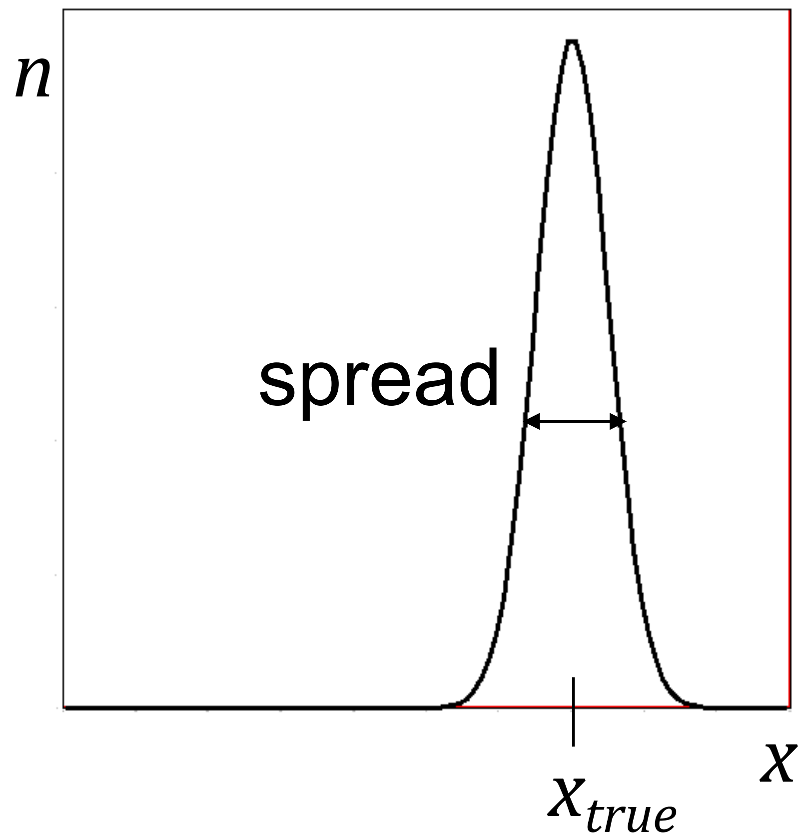


e.g. quiz scores

One entry (x) in this histogram
means one measurement (e.g. one score for every student)


(a)
1·4
1·6
1·8
2·0
(b)
1·2
0·8
0·4
n
(c)
120
80
40
n
(d)
120
80
40
n
(e)
n
h

Continuous distribution: infinite number of measurements



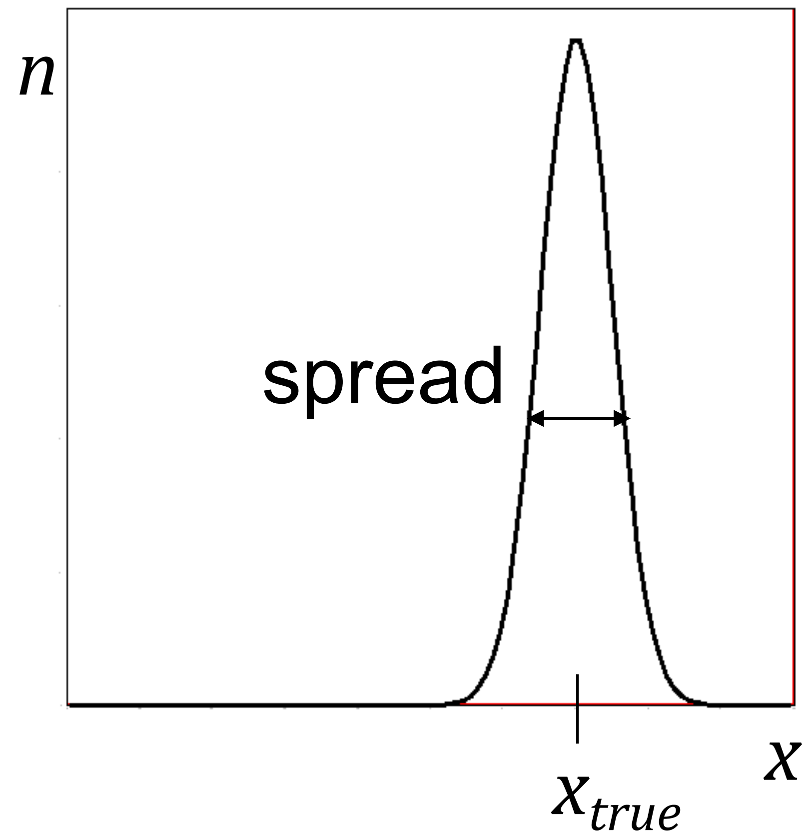
Types of experimental uncertainties: Random uncertainties

➔ statistical errors (arise from the inherent statistical nature of the phenomena being observed) and/or instrumental errors (arise from instrumental imprecisions)

➔ in a series of repeated measurements they produce slightly different values of the measured parameter x_{true}

➔ **may be handled by the theory of statistics**

Continuous distribution
(infinite number of measurements)

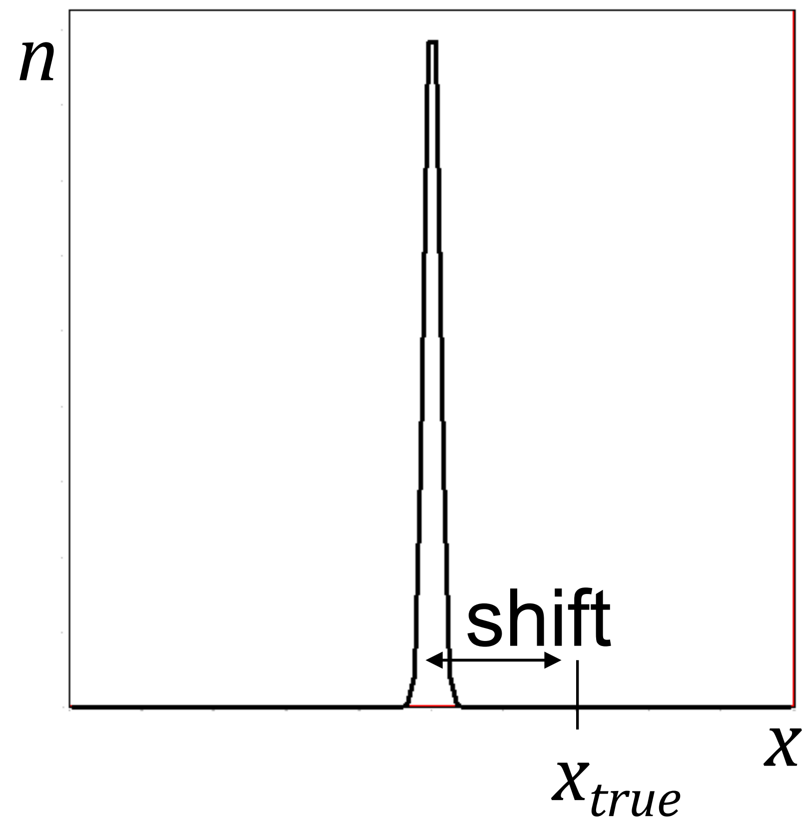


Types of experimental uncertainties
Systematic uncertainties

➔uncertainties in the bias of the data

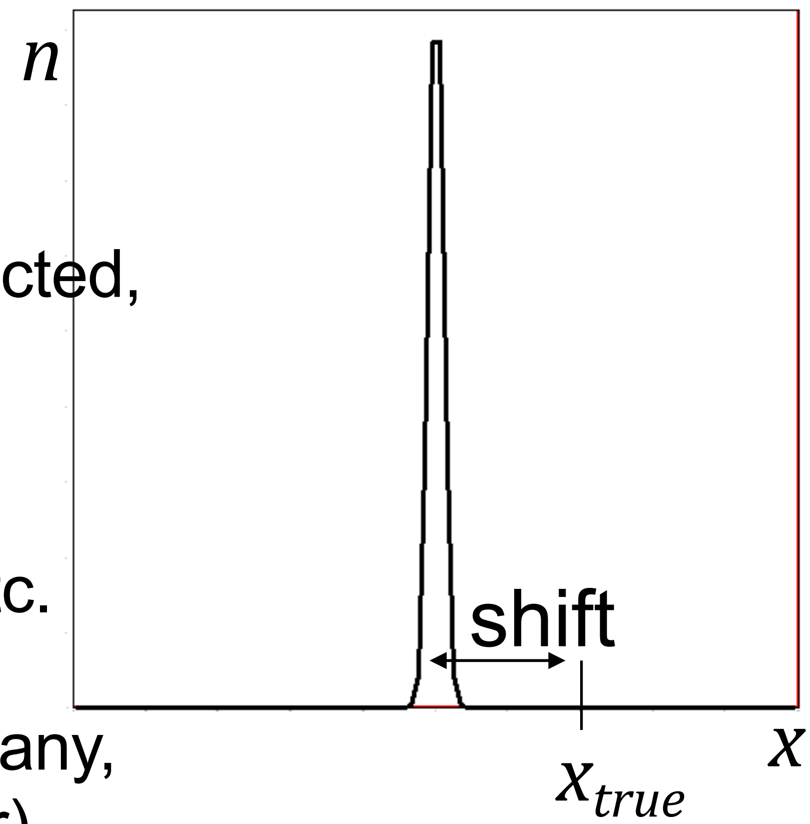
➔ in a series of repeated measure-ments they produce results that systematically shifted in the same direction by the same amount from the true value of the measured parameter

➔ difficult to identify the possible sources and estimate their magnitude.



Mistakes

➔Similar to systematic uncertainties in nature
➔Can be difficult to detect



Example1:
Writing 2.34 kHz instead of 2.43 kHz in your lab book. If not immediately corrected, will effect the precision of your result.

Other examples:
Misreading scales, confusion of units, etc.

Good experimentalist makes very few, if any, such mistakes (we'll not discuss it further)

Mistakes

The Gimli Glider Incident (1983), *from an article published in Soaring Magazine by Wade H.Nelson*

A Boeing 767 aircraft (Air Canada Flight 143) ran out of fuel mid-flight in 1983.
Reason: misunderstanding between metric and imperial units of volume.
The crew used 1.77 pounds per liter, instead of 0.8 kg per liter of kerosene.
(emergency landing in Gimli, Canada)

Mistakes

The New York Times

WORLD | U.S. | N.Y. / REGION | BUSINES

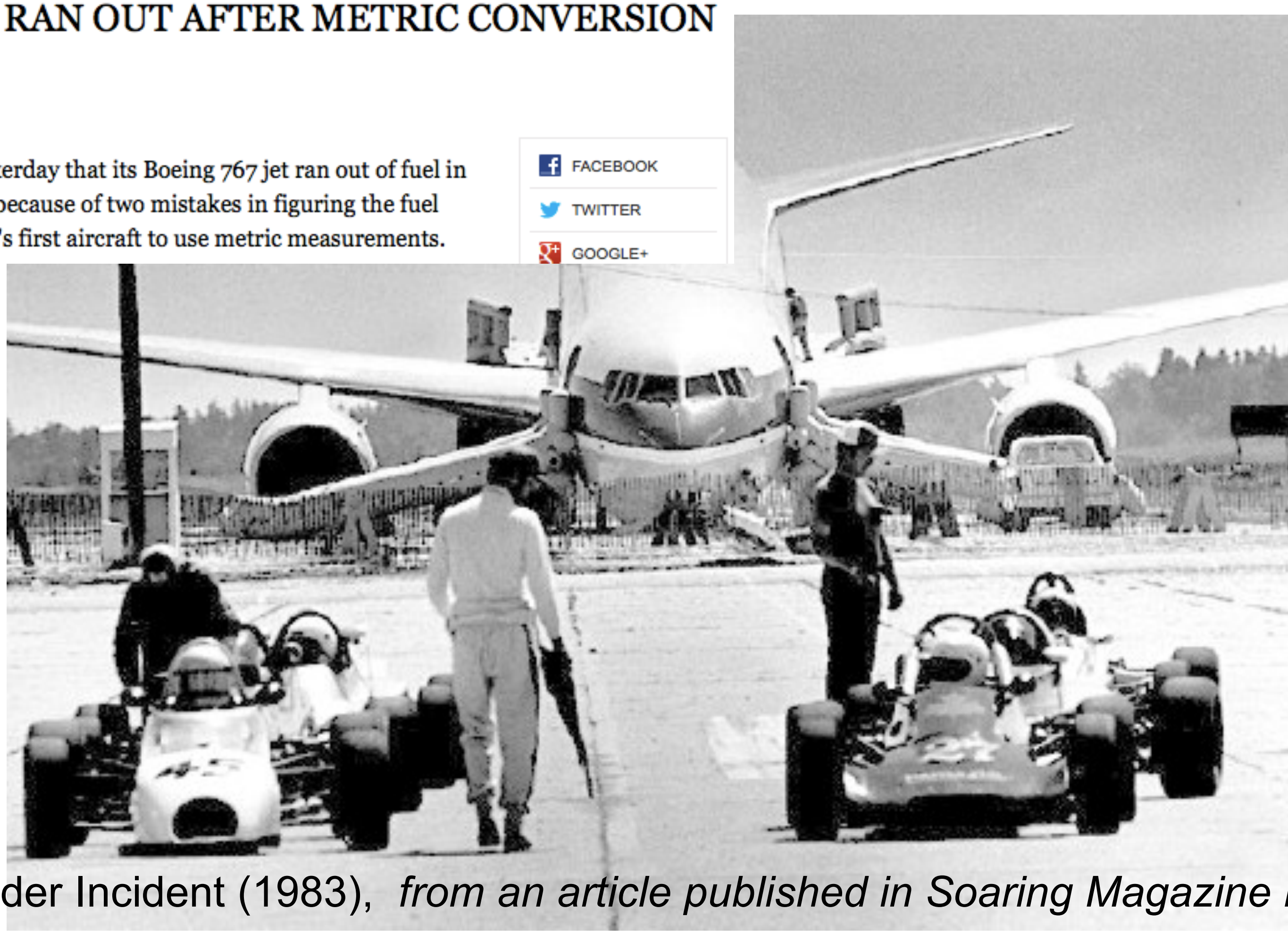
POLITICS EDUCATION TEXAS

JET'S FUEL RAN OUT AFTER METRIC CONVERSION ERRORS

By RICHARD WITKIN
Published: July 30, 1983

FACEBOOK
TWITTER
GOOGLE+

Air Canada said yesterday that its Boeing 767 jet ran out of fuel in midflight last week because of two mistakes in figuring the fuel supply of the airline's first aircraft to use metric measurements.



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Types of experimental uncertainties
Systematic uncertainties

➔uncertainties in the bias of the data

➔ in a series of repeated measure-
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➔ c
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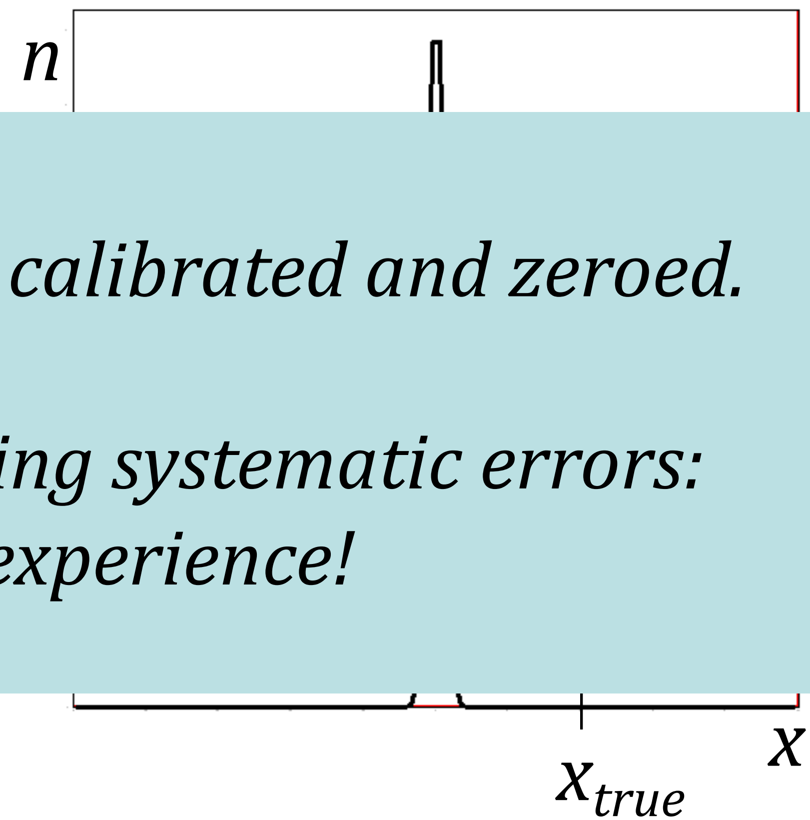
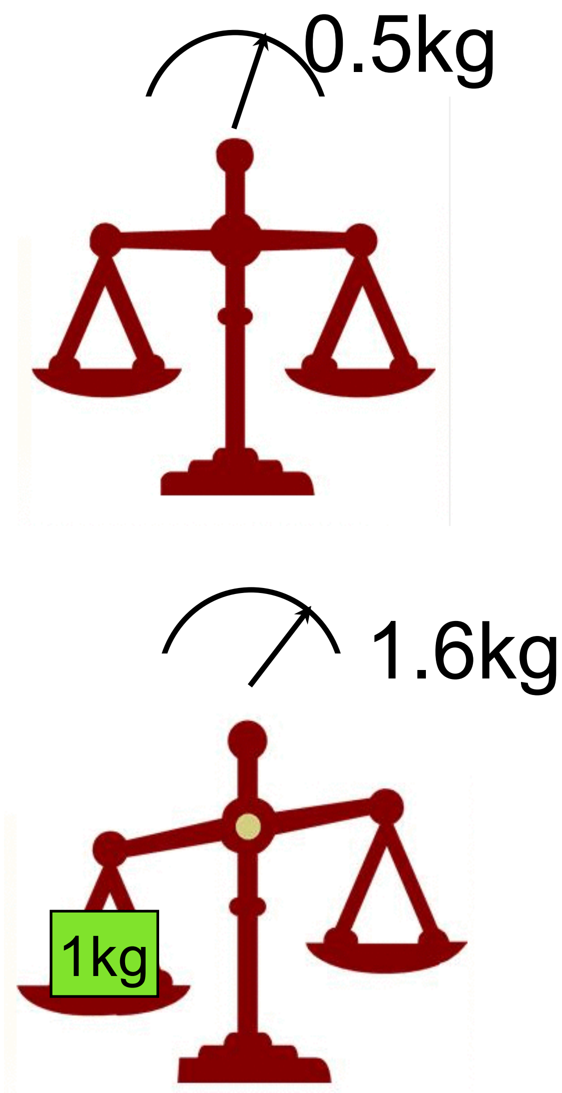
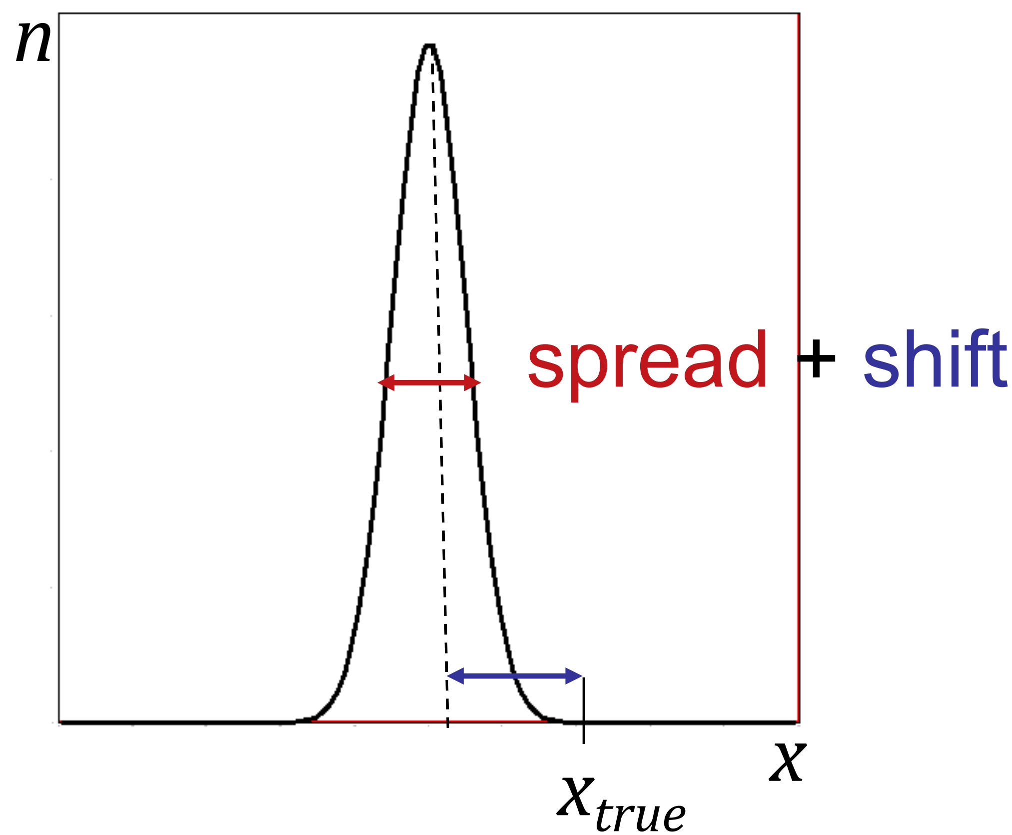


Ensure apparatus is properly calibrated and zeroed.

No simple rules for eliminating systematic errors: common sense + experience!

Most realistic situation:
Random and Systematic uncertainties





x can have a meaning of any measured quantity (e.g. box weight, acceleration due to gravity, etc)

A good experimental physicist:

R.Muller, "The Instant Physicist"

minimizes and realistically estimates the random errors of his/her apparatus

and

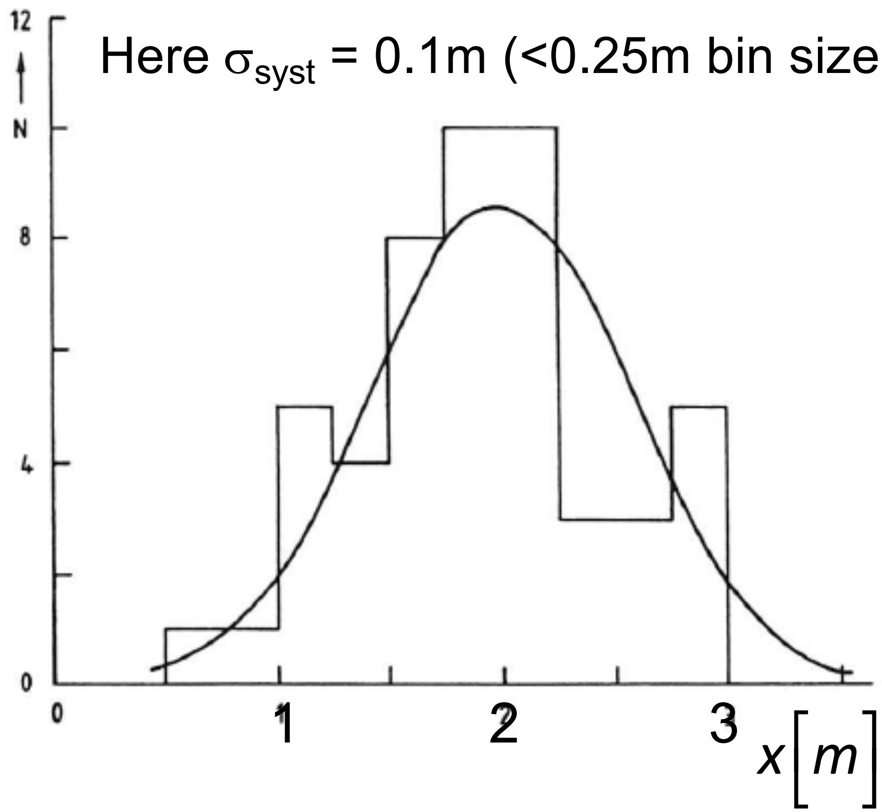
reduces the effect of systematic errors to much smaller levels.

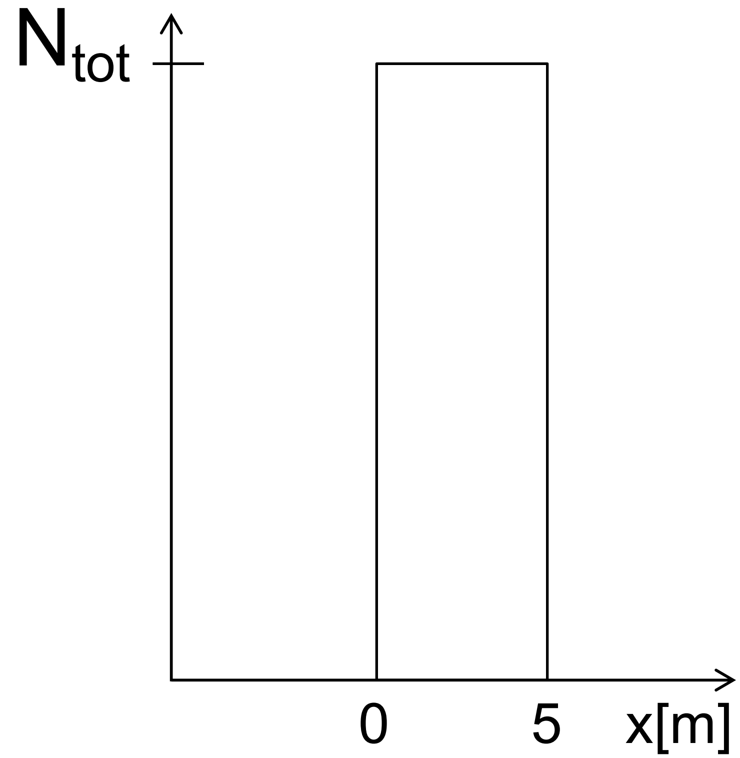
Multiple measurements: distribution

A) Random uncertainties dominate i.e. Measurement tool accuracy (systematic error) smaller than the bin size (device "calibrated" i.e. no offsets)

B) Random uncertainties much smaller than the measurement tool accuracy (systematic error)

Here σ_{syst} = 0.1m (<0.25m bin size)



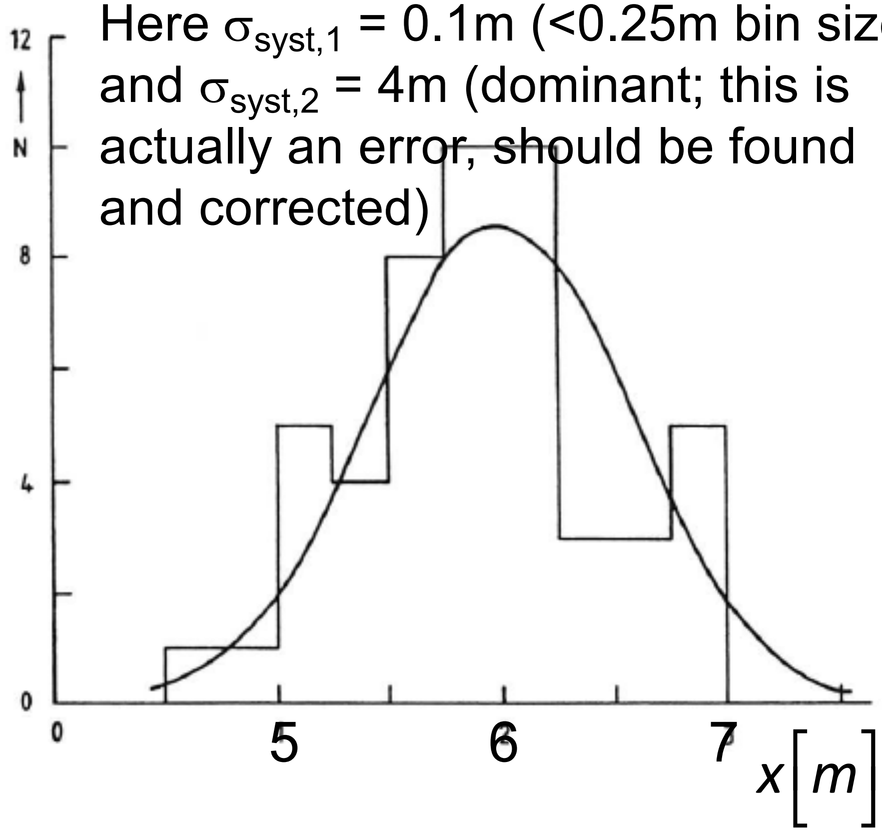


Histogram (finite number of measurements)
Total number of measurments:
N_{tot} = N (bin1)+N (bin2)+ ….
Function (infinite number of measurements)

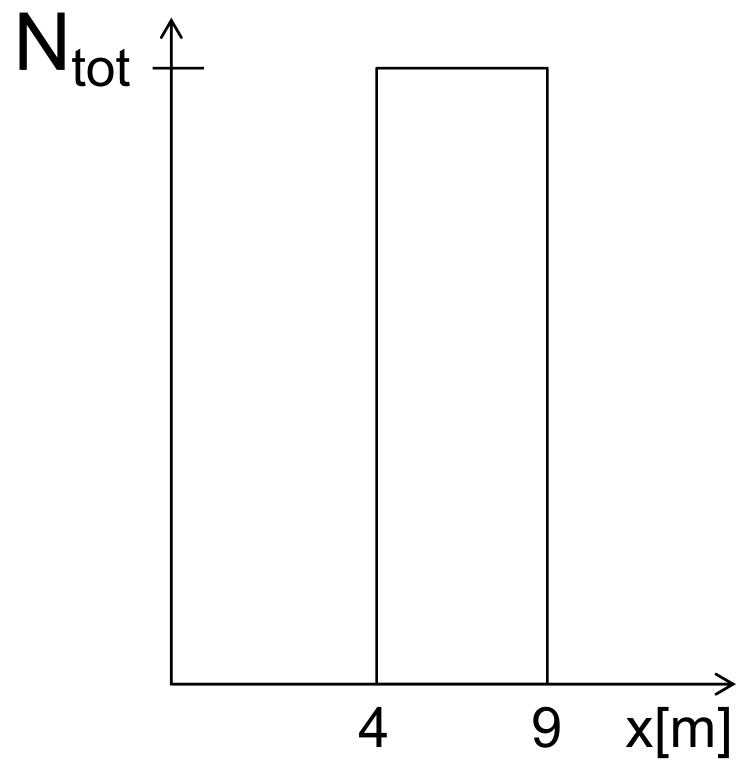
Multiple measurements: distribution

C) Random uncertainties dominate i.e. Measurement tool accuracy (systematic error) smaller than the bin size (device not “calibrated” i.e. there is an offset of 4m)

Here $\sigma_{syst,1}$ = 0.1m (<0.25m bin size) and $\sigma_{syst,2}$ = 4m (dominant; this is actually an error, should be found and corrected)



D) Random uncertainties much smaller than the measurement tool accuracy (systematic error)



Histogram (finite number of measurements)

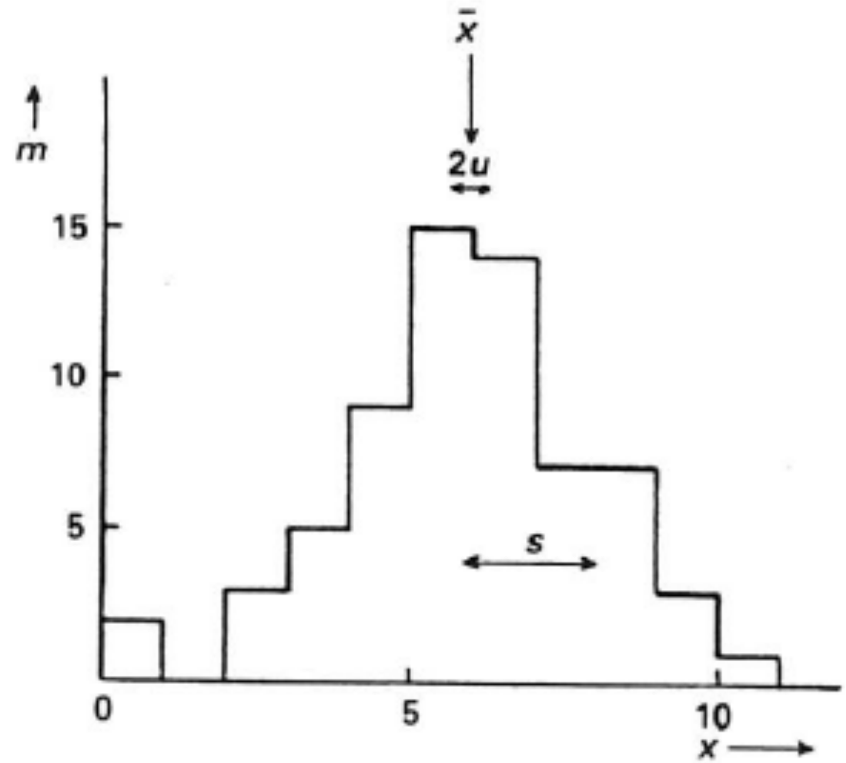
Total number of measurments:

N_{tot} = N (bin1)+N (bin2)+ ….

Function (infinite number of measurements)

Characteristic of the a distribution

- Sample mean μ
- Sample variance σ^2



One entry (x) in this histogram means one measurement

One measurement has an uncertainty of σ (we'll learn how to estimate it in Lecture2)
Result of an individual measurement

$$x \pm \sigma$$

How to present final experimental measurement results. Proper rounding.

Incorrect: (1.89999679 $\pm$ 0.00346) [m]

How to write it correctly?

1. Look at the uncertainty: 0.00**34**6 and then round to 2 most significant numbers. If the 3rd number is $\geq$ 5 then the 2nd significant number must be increased by 1, i.e. 0.00**34**6 ~ 0.0035.

2. Rounding the result is now straightforward:

Correct: (1.9000 $\pm$ 0.0035) [m]

1.9000(35) [m]

(19000 $\pm$ 35) x10^{-4} [m]

19000 (35) x10^{-4} [m]

Note: if the uncertainty is 0.0035, then it does not make sense to keep as many numbers in the measured value as possible (e.g. as your calculator displays), since 1.8999**9679** numbers marked in purple are not significant.

Rounding:

Lab reports: points will be subtracted if final results are not rounded properly

Exercises:

A. (1.9 +/- 0.189) [m]
B. (1.89999679) +/- 0.189 [m]
C. (1.90 +/- 0.19) [m]
D. (1.9 +/- 0.2) [m]

E. (23.24555 +/- 2.234) [m]
F. (23.2 +/- 2.2) [m]
G. (23 +/- 2) [m]

H. (0.00012378 +/- 0.00000568) [m]
I. (0.0001238 +/- 0.0000057) [m]
J. (0.000124 +/- 0.000006) [m]
K. (1.24 +/- 0.06)x10^{-4} [m]
L. 1.24(6) x10^{-4} [m]

Which are correct and which are incorrect?

How to present final experimental measurement results. Proper rounding (important for PHY252)

Example

Incorrect rounding: (1.89999679 ± 0.00346) [m]

How to write it correctly?

1. Look at the uncertainty: 0.00**34**6 and then round to 2 most significant numbers. If the 3rd number is $\geq$ 5 then the 2nd significant number must be increased by 1, i.e. 0.00**34**6 ~ 0.0035.

2. Rounding the result is now straightforward:

Correct: **(1.9000 ± 0.0035) [m]**

1.9000(35) [m]

$(19000 \pm 35)\ x10^{-4}$ [m]

$19000\ (35)\ x10^{-4}$ [m]

Note: if the uncertainty is 0.0035, then it does not make sense to keep as many numbers in the measured value as possible (e.g. as your calculator displays), since 1.8999**9679** numbers marked in purple are not significant.

Rounding:

Lab reports: points will be subtracted if final results are not rounded properly

Group work:

A. (1.9 +/- 0.189) [m]
B. (1.89999679) +/- 0.189 [m]
C. (1.90 +/- 0.19) [m]
D. (1.9 +/- 0.2) [m]

E. (23.24555 +/- 2.234) [m]
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K. (1.24 +/- 0.06)x10^{-4} [m]
L. 1.24(6) x10^{-4} [m]

Which are correct and which are incorrect? Round properly incorrect ones.

Experiment, Outcome, Event and Probability

Definition	Example
An **experiment** is a situation involving chance or probability that leads to results called outcomes.	the experiment is spinning the spinner.
An **outcome** is the result of a single trial of an experiment.	The possible outcomes are landing on yellow, blue, green or red.
An **event** is one or more outcomes of an experiment.	One event of this experiment is landing on blue.
Probability is the measure of how likely an event is.	The probability of landing on blue is ?

In order to measure probabilities, mathematicians have devised the following formula for finding the probability of an event.

Probability Of An Event
P(A) = The Number Of Ways Event A Can Occur / The total number Of Possible Outcomes

$$0 \le P(A) \le 1$$

The probability of an event is the measure of the chance that the event will occur as a result of an experiment

http://www.mathgoodies.com/lessons/vol6/intro_probability.html

Probability Interpretations

"It is possible for an exp. physicist to spend a lifetime analyzing data without realizing that there are two different fundamental approaches to statistics."

L.Lyons

1. Relative frequency *(frequentism)*

   A, *B* ... are outcomes of a repeatable experiment

$$P(A) = \lim_{n\to\infty} \frac{\text{times outcome is } A}{n}$$

Common in experimental physics (this course)

e.g. particle scattering, radioactive decay... (most useful in HEP)

2. Subjective probability *(bayesianism)*

   A, *B*, ... are hypotheses (statements that are true or false)

$$P(A) = \text{degree of belief that } A \text{ is true}$$

can provide more natural treatment of <u>non-repeatable</u> phenomena:
e.g. systematic uncertainties, probability that Higgs boson exists ...

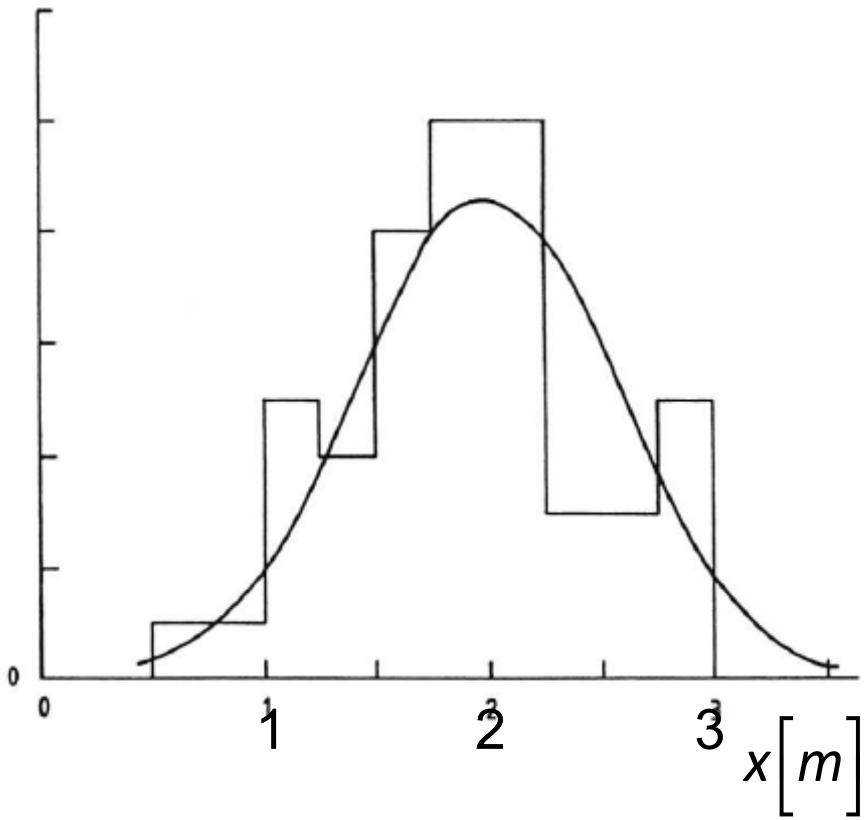
Measurement and Probability Distributions

- Measurement is a random process described by an abstract probability distribution whose parameters contain the information desired.
- The results of a measurement are then samples from this distribution which allow an estimate of the theoretical parameters.

Now we'll explain what this means ….

Probability distribution functions, expectation values and moments

Multiple measurements: distribution



Continuous line is a known function (typically Gaussian) so called Probability Distribution Function (PDF)

Probability Distribution Functions

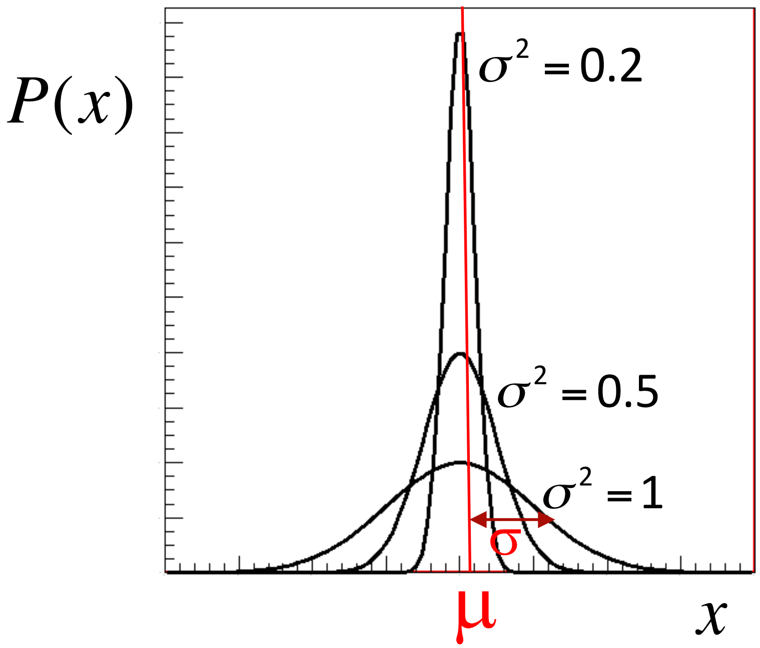
Many pdf' s, large number of problems in physics are described by a small number of theoretical distributions:

	Distribution	Example
→	Gaussian	Measurement error
→	Poisson	Number of events found
→	χ^2	Goodness-of-fit

Poisson, Gaussian PDF's – most common in experimental physics

The Gaussian Distribution

The Gaussian (also called "normal") pdf plays a central role in all of statistics and is the most ubiquitous distribution in all the sciences. Even in cases where its application is not strictly correct, **the Gaussian often provides a good approximation to the true pdf's**. It is defined as:



$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Described by two parameters: μ, σ

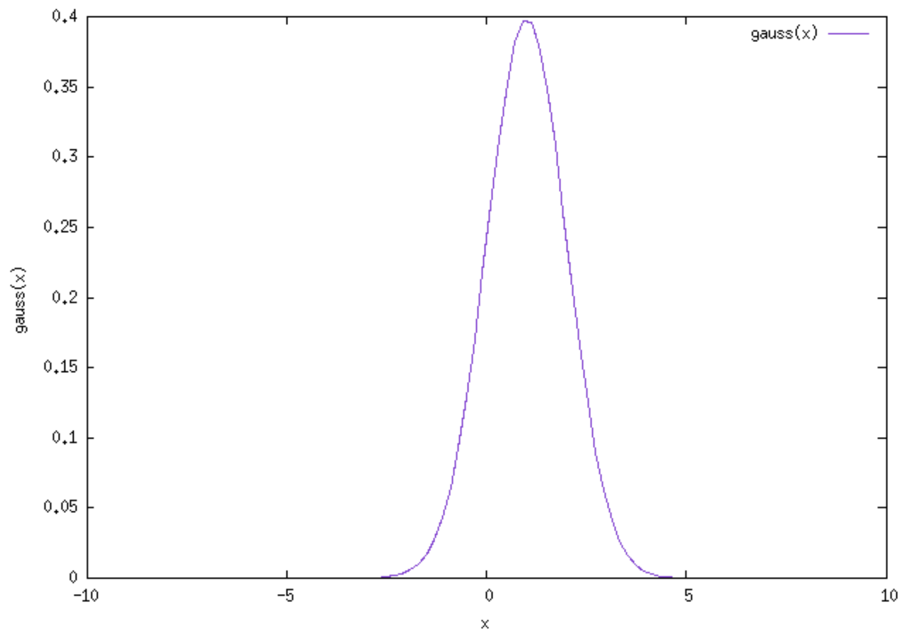
Expectation value (mean): $E[x] = \mu$

Variance: $V[x] = \sigma^2$

Standard deviation: σ

Example:

```
[gnuplot>
[gnuplot> PI=3.14; s=1; m=1;
[gnuplot> gauss(x)=1./(2*PI*s**2)**0.5*exp(-(x-m)**2/(2*s**2))
[gnuplot> set xlabel 'x'
[gnuplot> set ylabel 'gauss(x)'
[gnuplot> plot gauss(x)
 gnuplot>
```



Assignment (do @ home)

Using gnuplot, plot P(x) = Gaussian functions with μ=0.1 and
a) σ^2 =0.2
b) σ^2 =0.5
c) σ^2 =1
parameters.

Characteristics of Probability Functions

Random processes: described by the probability density function (pdf) which gives the expected frequency of occurrence for each possible outcome (random variable x).

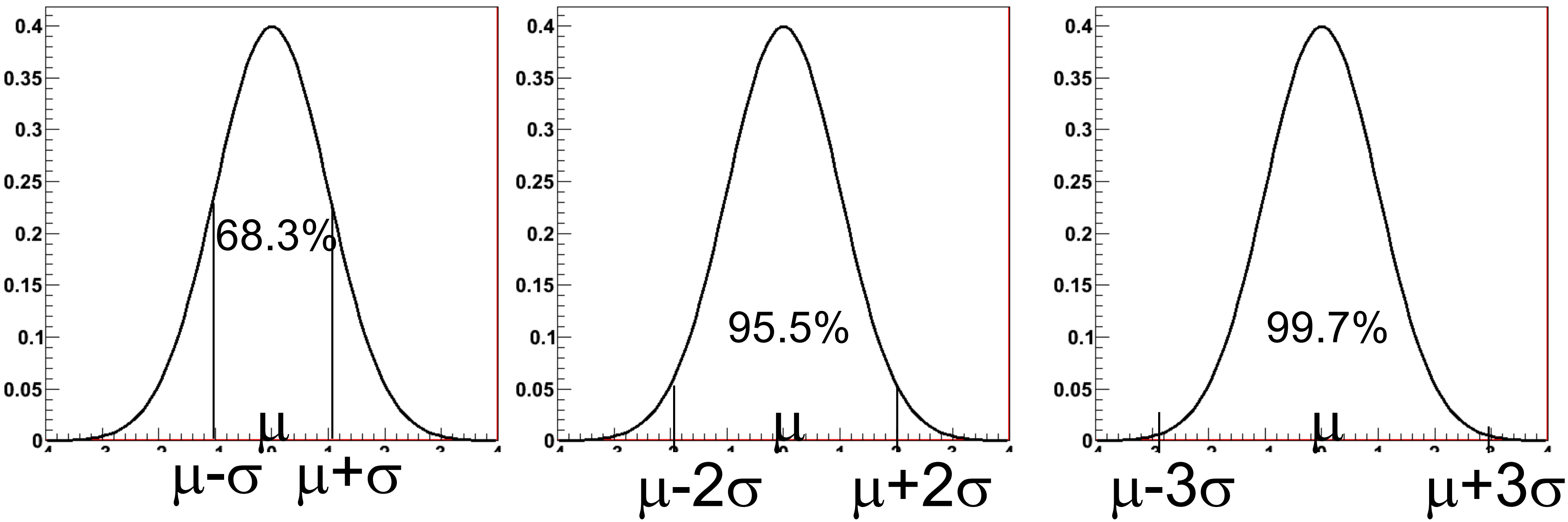
Example:

The process = throwing a single die, then x={1,…,6} and P(x)=1/6

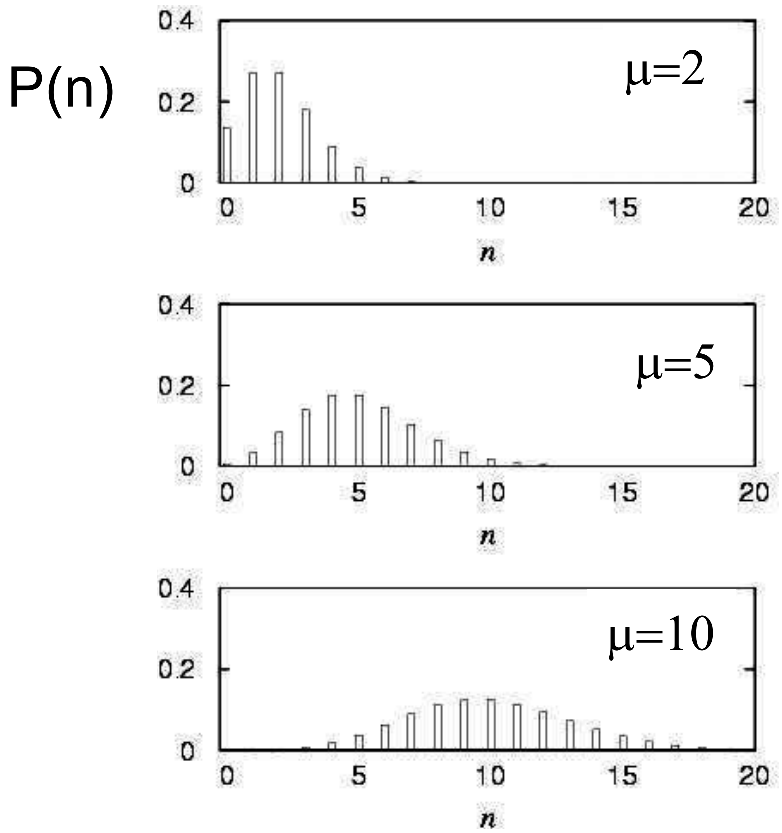
The random variable is then said to be distributed as P(x).

Random variable	PDF	Integral $P(x_1 \leq x \leq x_2)$	Normalization
Discrete	$P(x_i)$: *frequency at each* x_i	$= \sum_i P(x_i)$	$\sum_i P(x_i) = 1$
Continuous	P(x): *probability of finding x in interval x' to x' +dx' is P(x')dx'*	$= \int_{x_1}^{x_2} P(x)dx$	$\int P(x)dx = 1$

The area under the Gaussian bewteen integral intervals of σ
- an important practical quantity



The Poisson Distribution



$$P(n) = \frac{\mu^n e^{-\mu}}{n!}$$

Described by one parameters: μ

Expectation value (mean): $E[n] = \mu$

Variance: $V[n] = \mu$

Standard deviation: $\sigma = \sqrt{\mu}$

Example (important):

In all counting experiments, for *m* observed events (*m* is "large"), the standard deviation (i.e. uncertainty) is $\sqrt{m}$

G. Cowan, "Statistical Data Analysis"

Expectation Values, Distribution Moments

For a continuous random variable *x* with pdf P(x):

→ here we do not define or choose what the pdf P(x) is

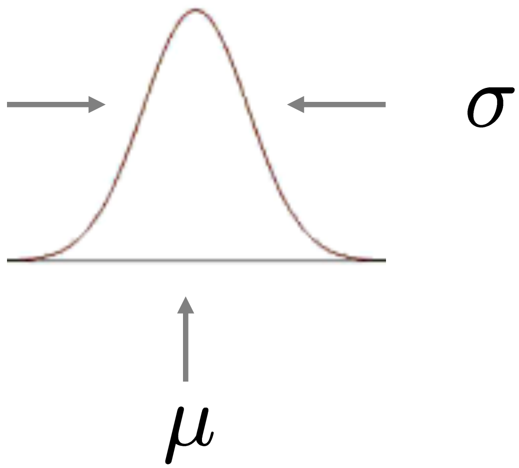
Expectation value of x: $E[x] = \int x\, P(x)\, dx$

The r-th moment of x about x_0: $E[(x - x_0)^r]$

- 1st moment about $x_0=0$

$$\mu = E[x] = \int x\, P(x)\, dx$$

(mean)



- 2nd moment about $x_0=\mu$ *(Variance)*

$$\sigma^2 = V[x] = E[x^2] - \mu^2 = E[(x-\mu)^2] = \int (x-\mu)^2\, P(x)\, dx$$

$\sigma = \sqrt{\sigma^2}$ *(Standard deviation)*

Multivariate distributions

For continuous random variables *x*,*y* with pdf P(x,y):

- *Means:*

$$\mu_x = E[x] = \iint xP(x,y)\,dxdy$$
$$\mu_y = E[y] = \iint yP(x,y)\,dxdy$$

- *Variances:*

$$\sigma_x^2 = E[x^2] - \mu_x^2 = E[(x-\mu_x)^2] = \iint (x-\mu_x)^2\,P(x,y)\,dxdy$$
$$\sigma_y^2 = E[y^2] - \mu_y^2 = E[(y-\mu_y)^2] = \iint (y-\mu_y)^2\,P(x,y)\,dxdy$$

Multivariate distributions, the covariance

For continuous random variables *x,y* with pdf P(x,y):

- *Covariance:*

- a measure of the linear correlation between the two variables.

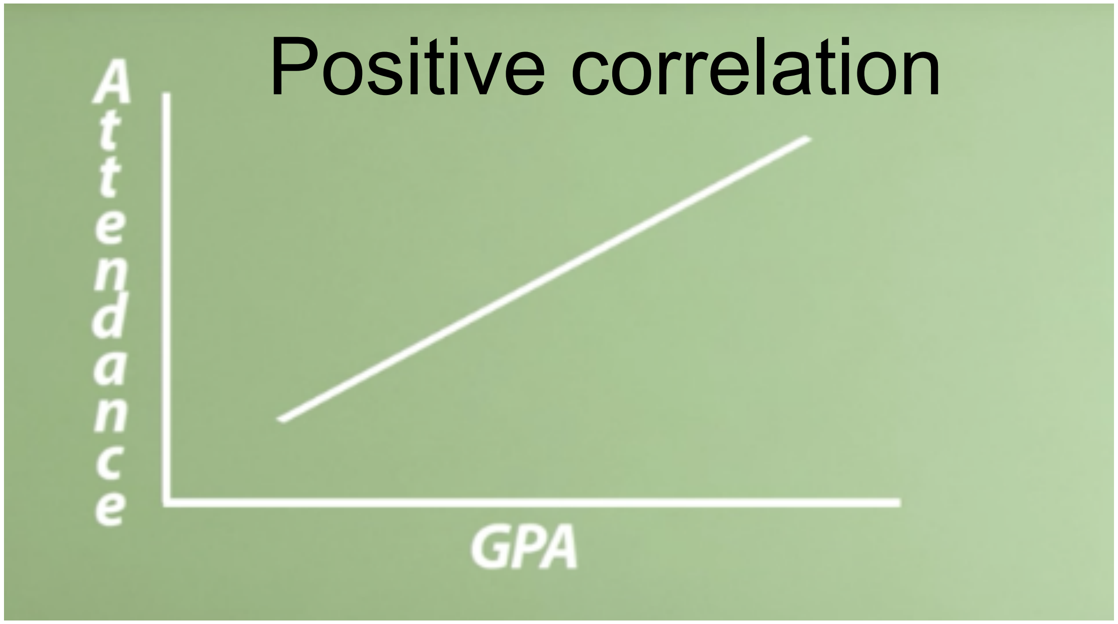
$$\mathrm{cov}(x,y) = E[xy] - \mu_x\mu_y = E[(x-\mu_x)(y-\mu_y)]$$
$$= \iint (x-\mu_x)(y-\mu_y)\, P(x,y)\, dxdy$$

- *Correlation coefficient:*

$$\rho_{xy} = \frac{\mathrm{cov}(x,y)}{\sigma_x\sigma_y} \qquad |\rho_{xy}| \le 1$$

Extra material (more advanced statistics/labs than PHY251/252)

Student	GPA	Days Present
1	4.00	180.0
2	2.50	150.0
3	4.00	170.0
4	3.90	180.0
5	3.75	177.0
6	3.80	180.0
7	2.90	140.0
8	3.10	169.0
9	3.25	168.0
10	3.40	152.0
11	3.30	150.0
12	3.90	170.0
13	1.35	109.0
14	4.00	180.0
15	1.00	108.0



Assumption: students pay attention and participate in the class activities.

Independent variables: no correlation

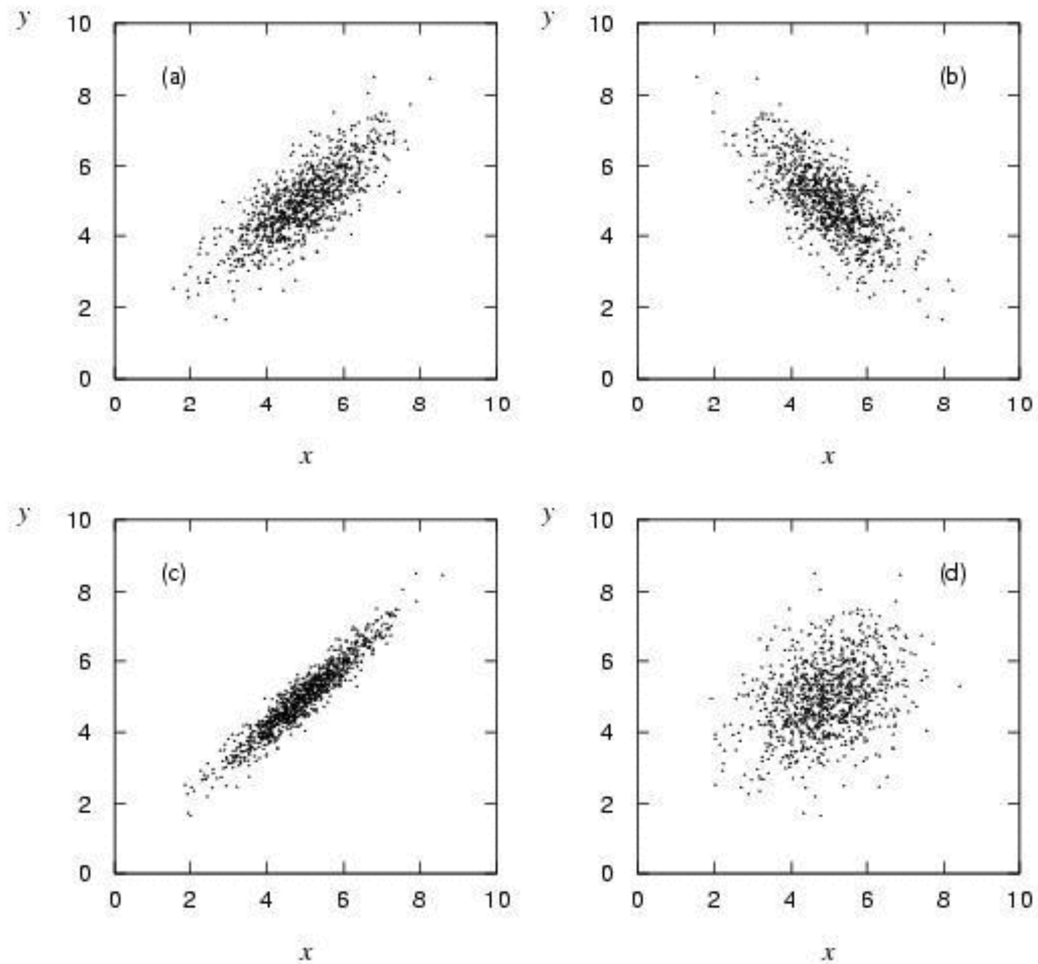
Extra material (more advanced statistics/labs than PHY251/252)

Correlation Coefficient - Examples

$$\rho_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} \qquad |\rho_{xy}| \leq 1$$

$\rho = 0.75$ $\rho = -0.75$

$\rho = 0.95$ $\rho = 0.25$



G. Cowan, "Statistical Data Analysis"

Sampling, sample moments and parameter estimation

Measurement is a random process described by an abstract probability distribution whose parameters contain the information desired. The results of a measurement are then samples from this distribution which allow an estimate of the theoretical parameters.

Sampling = experimental method by which information can be obtained about the parameters (like mean and variance) of an unknown distribution.

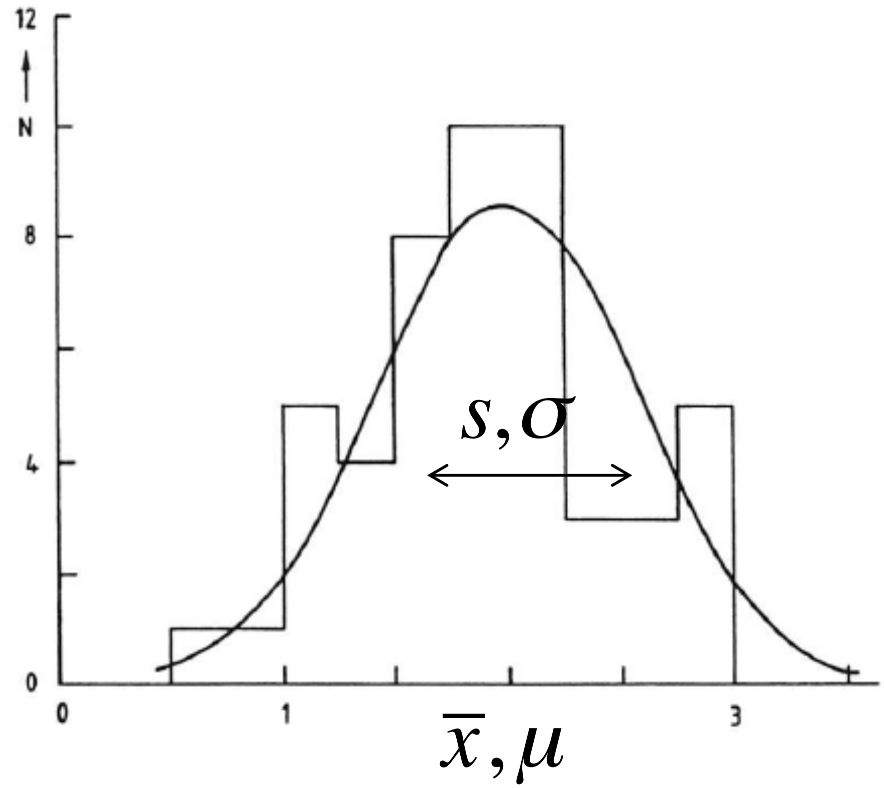
It is important to have a "representative" and "unbiased" sample.
Do NOT reject any data just because it does not "look right"!

You must find a reason for excluding the data (and only if a mistake cannot be corrected)

Characteristic of the a distribution

- Sample mean $\bar{x}$
  →estimate of true value μ

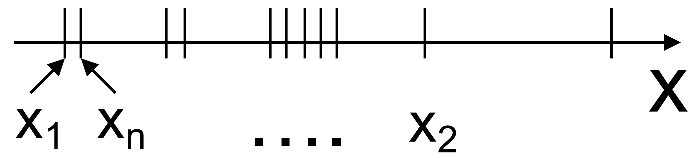
- Sample variance s^2
  →estimate of variance σ^2



Sample Moments

Let $x_1, x_2, \ldots, x_n$ be a sample of size n from a distribution with theoretical mean μ and variance σ^2 (both unknown).

- Sample mean (μ estimator):



$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- Sample variance (σ^2 estimator):

$$s^2 = \frac{1}{n}\sum (x_i - \mu)^2 = \frac{1}{n-1}\sum (x_i - \bar{x})^2 = s^2$$

- Sample variance on the mean:

$$u^2 = \frac{s^2}{n}$$

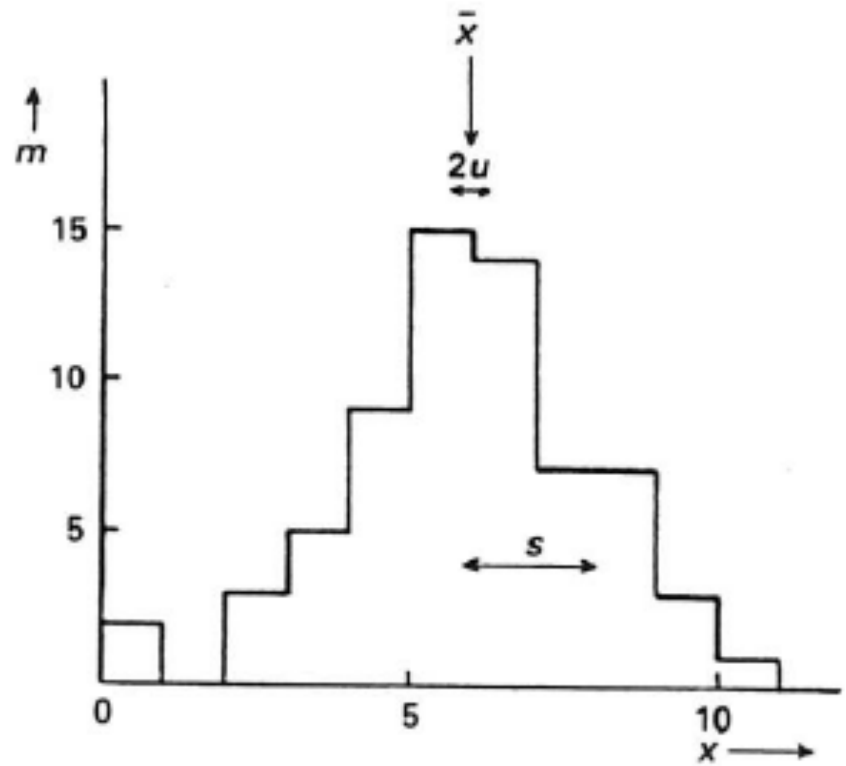
more data help to determine the mean to higher accuracy

Characteristic of the a distribution

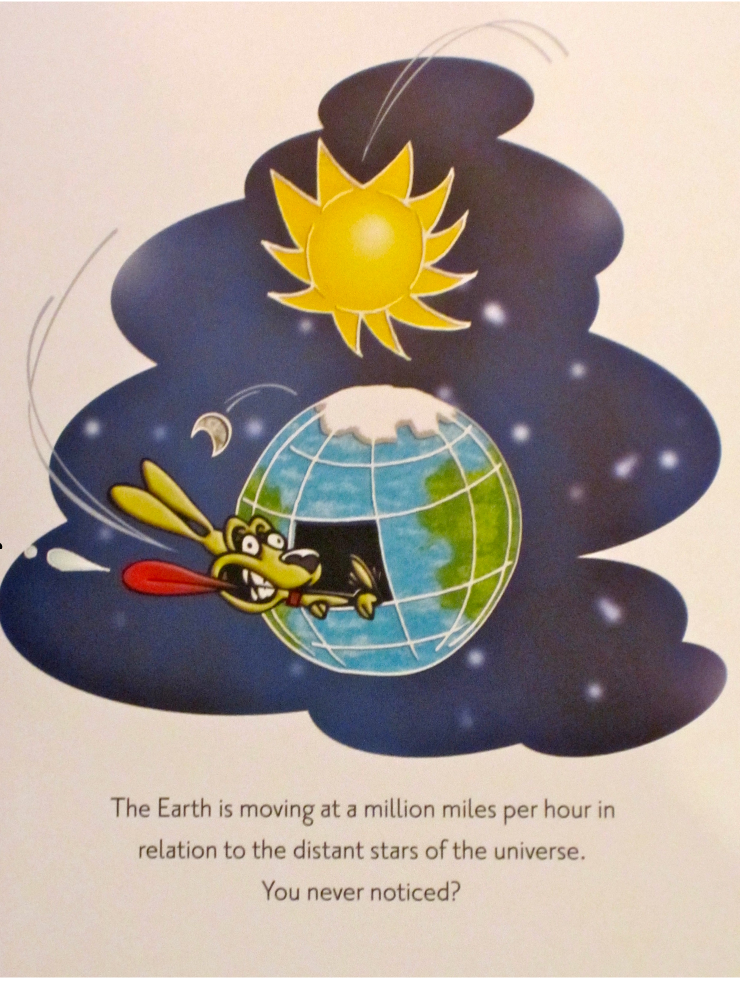
- Sample mean $\overline{x}$
  →estimate of true value μ

- Sample variance s^2
  →estimate of variance σ^2

s= uncertainty on a single measurement
u= uncertainty on the mean!



One entry (x) in this histogram means one measurement



R.Muller, “The Instant Physicist”

Example:

In an experiment consisting of 10 independent measurements, we measured the speed of Earth v_E in its revolution around the Sun and got the following results:

1. v_E= 29.7 [km/s]
2. v_E= 29.9 [km/s]
3. v_E= 29.9 [km/s]
4. v_E= 39.9 [km/s]
5. v_E= 29.8 [km/s]
6. v_E= 30.0 [km/s]
7. v_E= 39.7 [km/s]
8. v_E= 29.9 [km/s]
9. v_E= 29.8 [km/s]
10. v_E= 30.0 [km/s]

What is the best estimate (and its uncertainty) for v_E?
What is a single measurement uncertainty on v_E?

$$\bar{x} = \frac{1}{n}\sum_{i=n}^{n} x_i \quad =$$

$$= \frac{1}{10}\left(29.7 + 29.9 + 29.9 + 29.9 + 29.8 + 30.0 + 29.7 + 29.9 + 29.8 + 30.0\right)[\mathrm{km/s}]$$

$$= 29.853394 \text{ [km/s]}$$

$$s^2 = \frac{1}{n-1}\sum\left(x_i - \bar{x}\right)^2 = \frac{1}{9}[\left(29.7\text{-}\bar{x}\right)^2 + \left(29.9\text{-}\bar{x}\right)^2 + \left(29.9\text{-}\bar{x}\right)^2 + \left(29.9\text{-}\bar{x}\right)^2 + \left(29.8\text{-}\bar{x}\right)^2 +$$

$$+ \left(30.0\text{-}\bar{x}\right)^2 + \left(29.7\text{-}\bar{x}\right)^2 + \left(29.9\text{-}\bar{x}\right)^2 + \left(29.8\text{-}\bar{x}\right)^2 + \left(30.0\text{-}\bar{x}\right)^2] \text{ [km}^2/\text{s}^2]$$

$$= 0.009456 \text{ [km}^2/\text{s}^2]$$

$$u^2 = \frac{s^2}{n} = \frac{0.009456}{10}[\mathrm{km}^2/\mathrm{s}^2] = 0.0009456 \text{ [km}^2/\text{s}^2]$$

$$u = 0.030751 \text{ [km/s]} \approx 0.03[\mathrm{km/s}]$$

Result: $\bar{v}_E \pm \sigma_{vE} = \bar{x} \pm u = \left(29.85 \pm 0.03\right)[\mathrm{km/s}]$

@home Use Excel and make your computer do all the work for you!

Note: a single measurement has an uncertainty of s=sqrt(s^2) (not u!)

each measurement from the previous page e.g. v_E= 29.8 +/- 0.1 [km/s]

The meaning of sigma

Example

We measure the lifetime of the neutron:

$$\tau \pm \sigma_\tau = 950 \pm 20 [s]$$

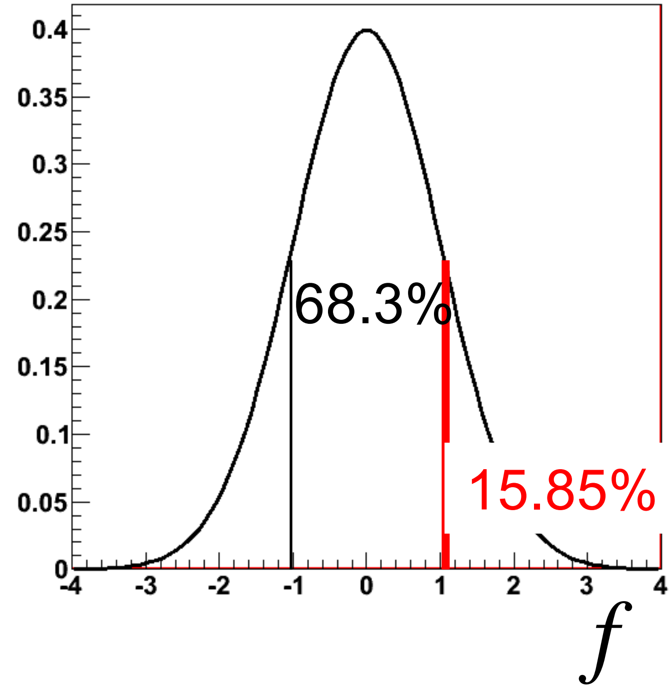
A certain theory predicts:

$$\tau_{th} = 910 [s]$$

To what extend are these numbers in agreement?

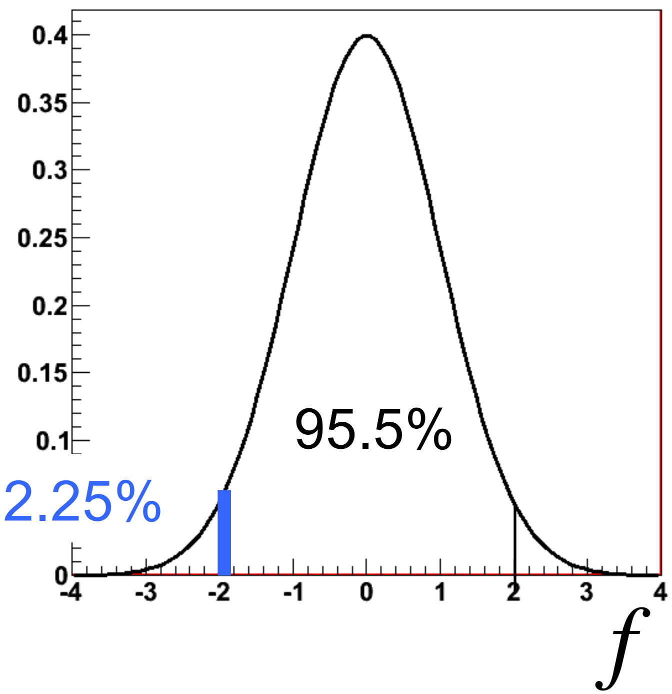
Recall:

The meaning of sigma



$$f = \frac{x-\mu}{\sigma}$$

e.g. if $x = \mu + \sigma \Rightarrow f = 1$



e.g. if $x = \mu - 2\sigma \Rightarrow f = -2$

The meaning of sigma

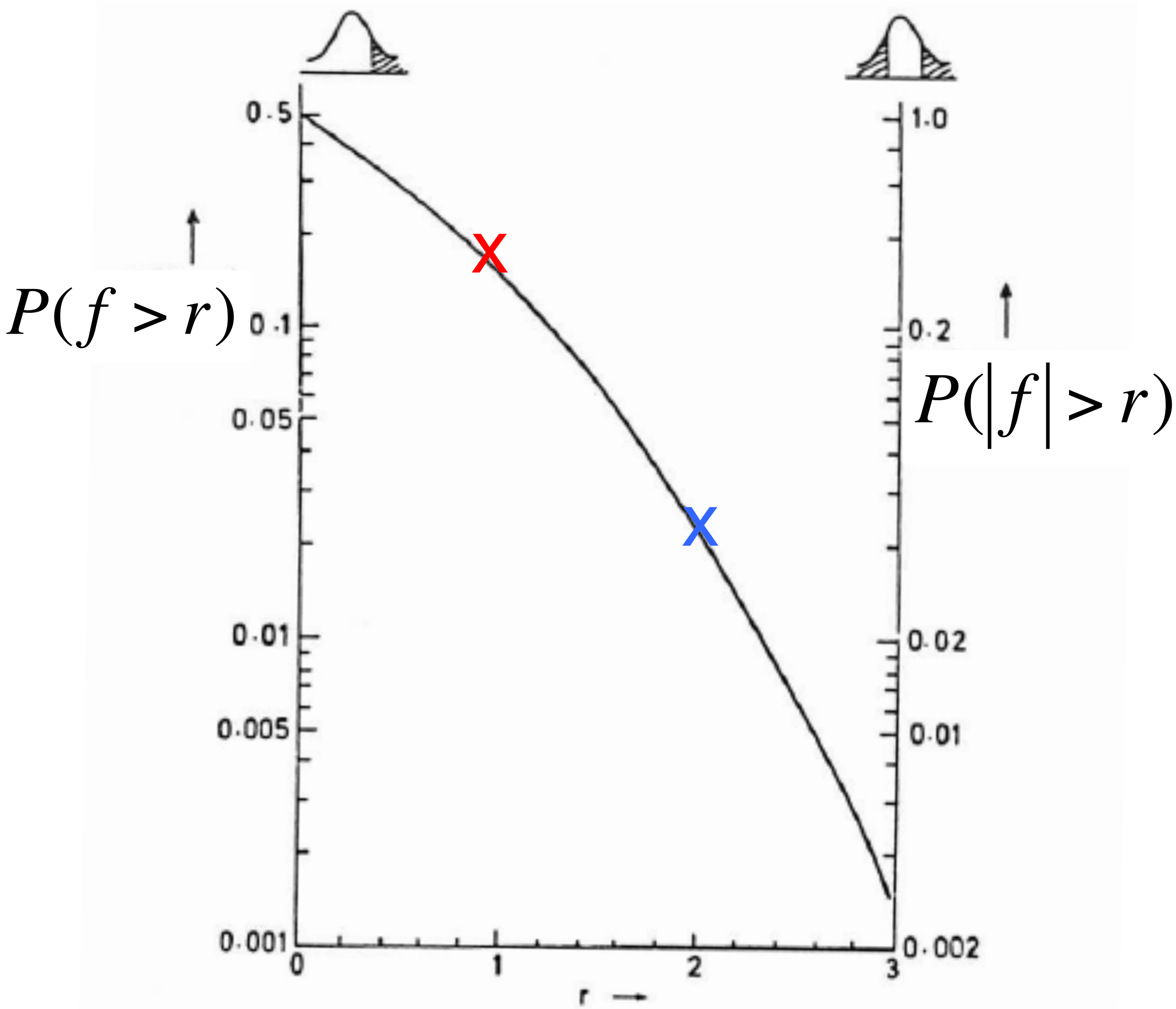


Fig. 1.7. The fractional area in the tails of a Gaussian distribution, i.e. the area with f greater than some specified value r, where f is the distance from the mean, measured in units of the standard deviation. The scale on the left hand vertical axis refers to the one-sided tail, while the right hand one is for $|f|$ larger than r. Thus for $r = 0$, the probabilities are $\frac{1}{2}$ and 1 respectively.

$$f = \frac{x - \mu}{\sigma}$$

The meaning of sigma

Example

We measure the lifetime of the neutron:

$$\tau \pm \sigma_\tau = 950 \pm 20 \left[s \right]$$

A certain theory predicts:

$$\tau_{th} = 910 \left[s \right]$$

To what extend are these numbers in agreement?

$$f = \frac{\tau - \tau_{th}}{\sigma_\tau} = \frac{40}{20} = 2$$

$$\left(\tau - \tau_{th} = 2\sigma_\tau \right)$$

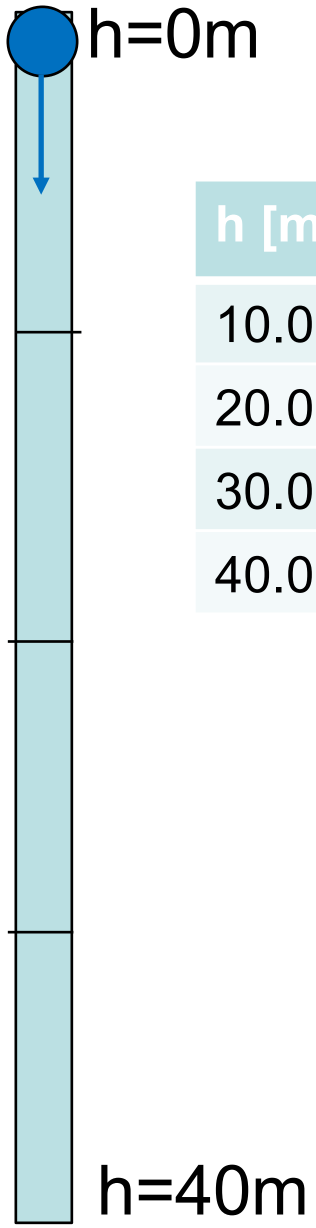
"2 sigma difference"

Interpretation: The corresponding probability is 1-0.955=0.046 i.e. 4.6%.
If 1000 experiments of the same precision as ours are performed, if the theory is correct, and if there are no biases, then results from 46 experiments will differ from the theoretical value by at least as much as ours does.

Error propagation

Example2: $h(t) = gt^2/2$

Evaluate g, and its uncertainty σ_g, assuming we measured h and t (4 measurements tota
and we know the precision of h and t to be σ_h =0.01m and σ_t = 0.01s respectively.



Assume h and t are uncorrelated.

h [m]	t [s]
10.00 m +/- 0.01m	1.43 s +/- 0.01s
20.00 m +/- 0.01m	2.02 s +/- 0.01s
30.00 m +/- 0.01m	2.47 s +/- 0.01s
40.00 m +/- 0.01m	2.86 s +/- 0.01s

$g \pm \sigma_g$?

Error propagation

Suppose we measured a set of e.g. n variables: $x_1, x_2, \ldots x_n$. with uncertainties σ_{x1}, σ_{x2} $\ldots \sigma_{xn}$ and covariances $\mathrm{cov}(x_1,x_2)$, … $\mathrm{cov}(x_{n-1},x_n)$. Consider a function $f=f(x_1,x_2,\ldots x_n)$. What is the variance of f i.e. $(\sigma_f)^2$ (f is determined from $x_1,x_2,\ldots x_n$; we want to know what's the uncertainty on f knowing uncertainties on $x_1,x_2,\ldots x_n$ and their covariances)

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2 +$$

$$+2\left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial f}{\partial x_2}\right)\mathrm{cov}(x_1,x_2) + \ldots\ldots + 2\left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial f}{\partial x_n}\right)\mathrm{cov}(x_1,x_n) + \ldots\ldots + 2\left(\frac{\partial f}{\partial x_{n-1}}\right)\left(\frac{\partial f}{\partial x_n}\right)\mathrm{cov}(x_{n-1},x_n)$$

Special case: if $x_1,x_2,\ldots x_n$ are uncorrelated **(THIS CLASS):**
$\mathrm{cov}(x_1,x_2)=0$, … $\mathrm{cov}(x_{n-1},x_n)=0$, then the variance of f is:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2$$

The most general case, this formula works for all functions and small uncertainties.

Error propagation

Example1: f=f(x,y,z), where x,y and z are uncorrelated.
The variance of f is:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2$$

Note: if there are more then 3 variables which are measured, one should add more terms in above equations. If there are less than 3 variables (e.g. only x and y are measured, one should remove all terms with z variable in above equations).

Combining Uncorrelated Errors: Special cases

Let f = f(x, y) and variables x, y are uncorrrelated

$$\sigma_x, \sigma_y - known, \mathrm{cov}(x,y) = 0$$

- Linear case:

$$f = x \pm y$$

$$\sigma_f^2 = \sigma_x^2 + \sigma_y^2$$

← absolute errors are relevant

- Products

$$f = x^a y^b$$

$$\left(\frac{\sigma_f}{f}\right)^2 = a^2\left(\frac{\sigma_x}{x}\right)^2 + b^2\left(\frac{\sigma_y}{y}\right)^2$$

$$f = xy, \; f = x/y$$

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

Fractional errors are relevant and must be small !
(for larger errors, use a numerical method)

Error propagation

Example2: $h(t) = gt^2/2$

Evaluate g, and its uncertainty σ_g, assuming we measured h and t (4 measurements tota
and we know the precision of h and t to be σ_h =0.01m and σ_t = 0.01s respectively.
Assume h and t are uncorrelated.

$$g(h,t) = 2ht^{-2}$$

Variance: $$(\sigma_g)^2 = \left(\frac{\partial g}{\partial h}\right)^2 (\sigma_h)^2 + \left(\frac{\partial g}{\partial t}\right)^2 (\sigma_t)^2$$

$$\frac{\partial g}{\partial h} = 2t^{-2}$$

$$\frac{\partial g}{\partial t} = 2h(-2)t^{-3} = -4h\,t^{-3}$$

$$(\sigma_g)^2 = (2t^{-2})^2 (\sigma_h)^2 + (-4h\,t^{-3})^2 (\sigma_t)^2$$

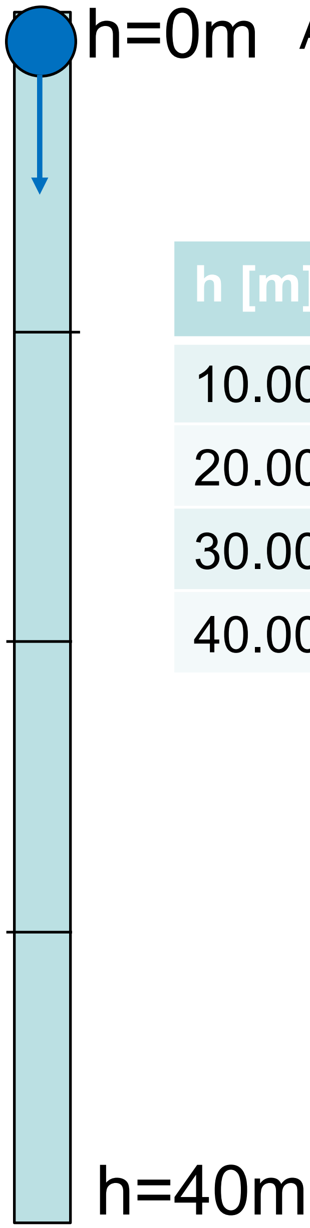
$$(\sigma_g)^2 = \left(\frac{4}{t^4}\right) (\sigma_h)^2 + \left(\frac{16h^2}{t^6}\right) (\sigma_t)^2$$

Standard deviation: $$\sigma_g = \sqrt{\left(\frac{4}{t^4}\right) (\sigma_h)^2 + \left(\frac{16h^2}{t^6}\right) (\sigma_t)^2}$$
(uncertainty)

Error propagation

Example2: $h(t) = gt^2/2$

Evaluate g, and its uncertainty σ_g, assuming we measured h and t (4 measurements total and we know the precision of h and t to be σ_h =0.01m and σ_t = 0.01s respectively.



Assume h and t are uncorrelated.

h [m]	t [s]	g+/- σ_g
10.00 m +/- 0.01m	1.43 s +/- 0.01s	
20.00 m +/- 0.01m	2.02 s +/- 0.01s	
30.00 m +/- 0.01m	2.47 s +/- 0.01s	
40.00 m +/- 0.01m	2.86 s +/- 0.01s	

Exercise @ home :
Fill this table out

Combining Results of Different Experiments

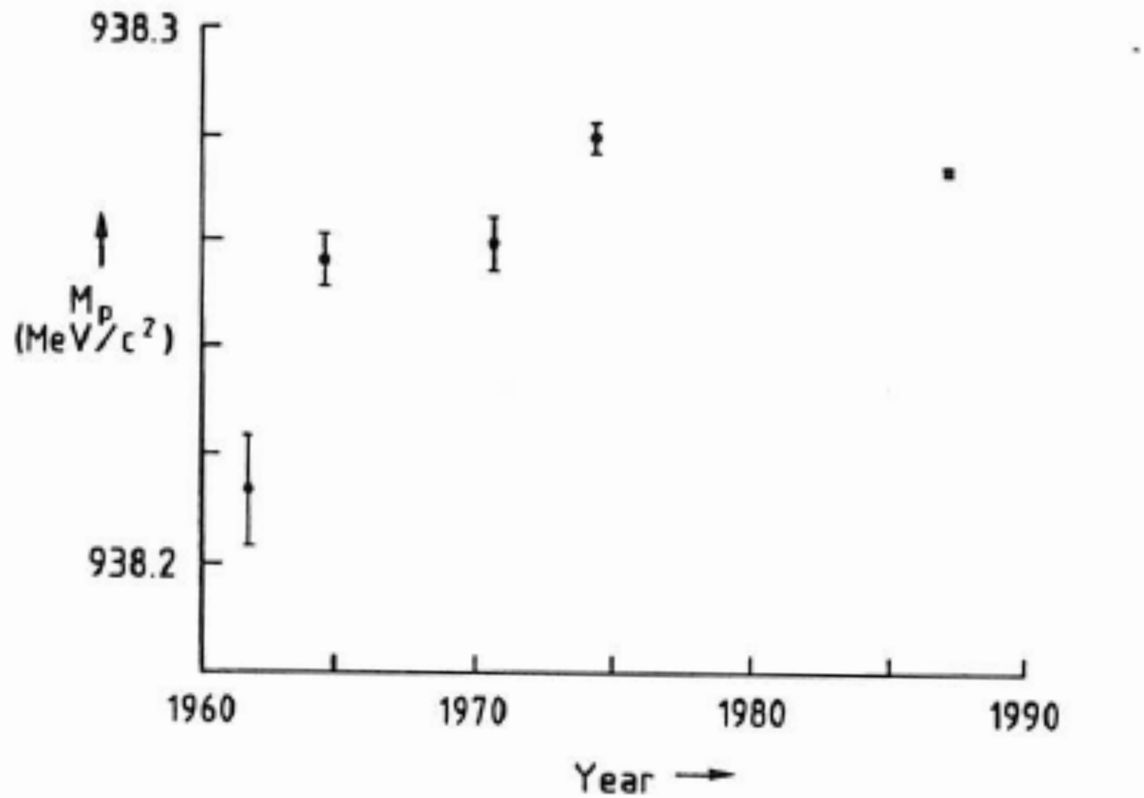


Fig. 1.11. The world average value of the proton mass M_p, as a function of time. The mass is quoted in MeV/c^2. In these units, the electron mass is 0.5109991 MeV/c^2, with an error of 2 in the last decimal place. (Based on information from the Particle Data Group.)

Combining Results of Different Experiments

When n experiments measure the same physical quantity and give a set of results a_i with different uncertainties σ_i^2, then the best estimate of a and its accuracy σ :

$$a = \frac{\sum_{i=1}^{n} \left(a_i / \sigma_i^2 \right)}{\sum_{i=1}^{n} \left(1 / \sigma_i^2 \right)} \qquad \sigma^2 = \frac{1}{\sum_{i=1}^{n} \left(1 / \sigma_i^2 \right)}$$

Each experiment is to be weighted by a factor $1/\sigma_{i.}$ In this approach we do not check the degree to which a_i are mutually consistent.

Exercise: calculate a when all experiments have the same accuracies (σ_i are the same)

$$a = \frac{\sum_{i=1}^{n} \left(a_i / \sigma_i^2 \right)}{\sum_{i=1}^{n} \left(1 / \sigma_i^2 \right)} \xrightarrow[\sigma_i^2 = const]{} \boxed{a = \frac{1}{n} \sum_{i=1}^{n} a_i} \qquad \sigma^2 = \frac{1}{\sum_{i=1}^{n} \left(1 / \sigma_i^2 \right)} \xrightarrow[\sigma_i^2 = const]{} \boxed{\sigma^2 = \frac{\sigma_i^2}{n}}$$

Least squares fitting

- Hypothesis testing
- Parameter fitting

Least squares fitting

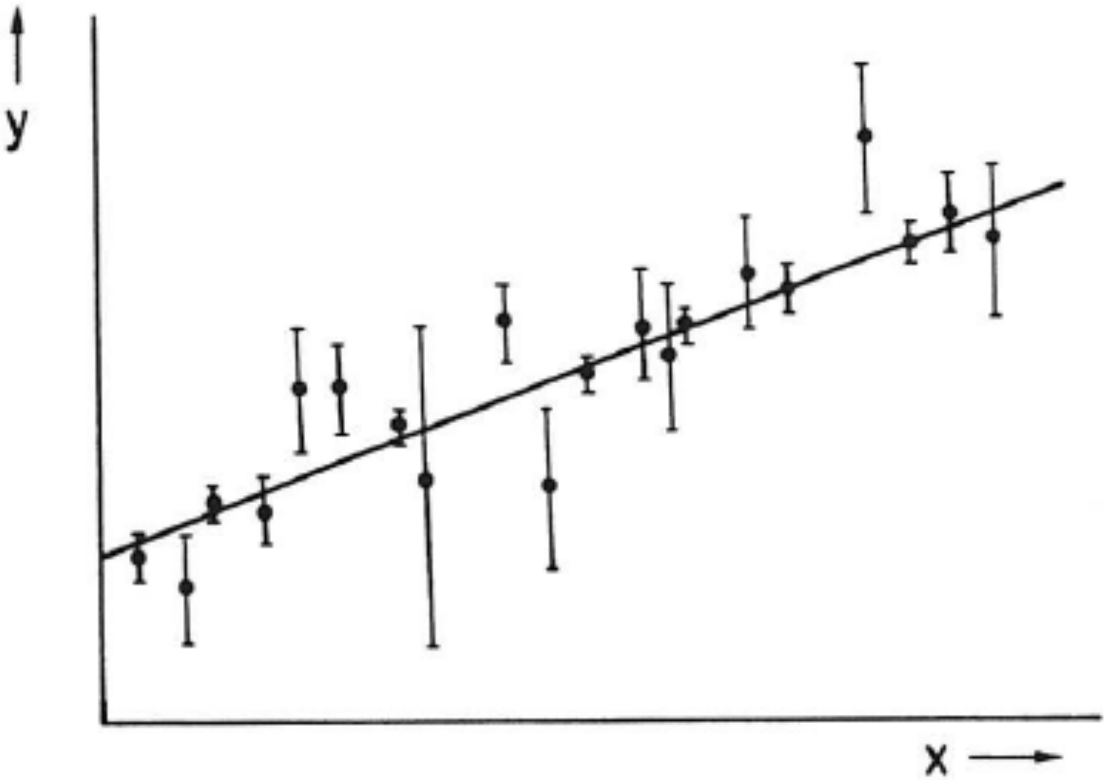


Table 2.1. *Possible fitting functions*
The set of data points y^{obs} is compared with the corresponding theoretical predictions y^{th} via eqn (2.1). Some possible examples of $y^{th}(x)$ are given, with the parameters involved in the theoretical predictions being shown explicitly.

Type	y^{th}	Parameters
Constant	c	c
Proportionality	mx	m
Straight line	$ax+b$	a, b
Parabolic	$a+bx+cx^2$	a, b, c
Inverse powers	$a+b/x+\cdots$	$a, b, \ldots$
Harmonic	$A\sin k(x-x_0)$	A, k, x_0
Fourier	$\sum a_n \cos nx$	$a_0, a_1, a_2, \ldots$
Exponential	$Ae^{\lambda x}$	A, λ
Mixed	$\begin{Bmatrix} F_1(x,\alpha_1), x \le c \\ F_2(x,\alpha_2), x > c \end{Bmatrix}$	α_1, α_2, c

- want a weighted fit
- **don't** use plotting tool from 1st year labs! (it doesn't use uncertainties of individual datapoints → not a weighted fit!)

The Least Square Method

Suppose we measured n points at x_i and got results: $f_i \pm \sigma_{i,f}$
We want to fit a function g to these data $g(x_i; a_1, a_2, ... a_m)$ where $a_1, a_2, ... a_m$ are unknown parameters to be determined and m<n.

The method of least squares (also called as chi-square χ^2 minimalization) states that the best values of a_j are those for which the sum:

$$S = \sum_{i=1}^{n} \left[\frac{f_i - g(x_i; a_j)}{\sigma_{i,f}} \right]^2 \longrightarrow \chi^2$$

is a minimum.

If f_i is Gaussian distributed with mean $g(x_i,a_j)$ and variance $(\sigma_{I,f})^2$

This method is general and does not require parent distributions.

To find a_j one must solve the system of equations $\frac{\partial S}{\partial a_j} = 0$

Depending on the function g(x), equation may or may not yield on analytic solution. In general, numerical methods must be used to minimize S.

Linear Fits. The straight line.

Let's consider a function: g(x)=ax+b, where the parameters parameters a and b are to be determined. The function S is:

$$S = \sum \frac{(f_i - ax_i - b)^2}{\sigma_{i,f}^2}$$

Taking partial derivatives:

$$\frac{\partial S}{\partial a} = -2\sum \frac{(f_i - ax_i - b)x_i}{\sigma_{i,f}^2} = 0$$

$$\frac{\partial S}{\partial b} = -2\sum \frac{(f_i - ax_i - b)}{\sigma_{i,f}^2} = 0$$

$$A \equiv \sum \frac{x_i}{\sigma_{i,f}^2} \qquad B \equiv \sum \frac{1}{\sigma_{i,f}^2}$$

$$C \equiv \sum \frac{f_i}{\sigma_{i,f}^2} \qquad D \equiv \sum \frac{x_i^2}{\sigma_{i,f}^2}$$

$$E \equiv \sum \frac{x_i f_i}{\sigma_{i,f}^2} \qquad F \equiv \sum \frac{f_i^2}{\sigma_{i,f}^2}$$

Linear Fits. The straight line.

g(x)=ax+b

$$2\left(-E + aD + bA\right) = 0$$

$$2\left(-C + aA + bB\right) = 0$$

$$\longrightarrow$$

$$a = \frac{EB - CA}{DB - A^2}$$

$$b = \frac{DC - EA}{DB - A^2}$$

Where A through F are determined from the data:

$$A \equiv \sum \frac{x_i}{\sigma_{i,f}^2} \qquad C \equiv \sum \frac{f_i}{\sigma_{i,f}^2} \qquad D \equiv \sum \frac{x_i^2}{\sigma_{i,f}^2}$$

$$B \equiv \sum \frac{1}{\sigma_{i,f}^2} \qquad E \equiv \sum \frac{x_i f_i}{\sigma_{i,f}^2} \qquad F \equiv \sum \frac{f_i^2}{\sigma_{i,f}^2}$$

Linear Fits. The straight line.

g(x)=ax+b

Not derived here

$$2\left(-E + aD + bA\right) = 0$$

$$2\left(-C + aA + bB\right) = 0$$

$\longrightarrow$

$$a = \frac{EB - CA}{DB - A^2}$$

$$b = \frac{DC - EA}{DB - A^2}$$

$$\sigma_a^2 = \frac{B}{DB - A^2}$$

$$\sigma_b^2 = \frac{D}{DB - A^2}$$

Where A through F are determined from the data:

$$A \equiv \sum \frac{x_i}{\sigma_{i,f}^2} \qquad C \equiv \sum \frac{f_i}{\sigma_{i,f}^2} \qquad D \equiv \sum \frac{x_i^2}{\sigma_{i,f}^2}$$

$$B \equiv \sum \frac{1}{\sigma_{i,f}^2} \qquad E \equiv \sum \frac{x_i f_i}{\sigma_{i,f}^2} \qquad F \equiv \sum \frac{f_i^2}{\sigma_{i,f}^2}$$

Special case: g(x)=ax+**b**

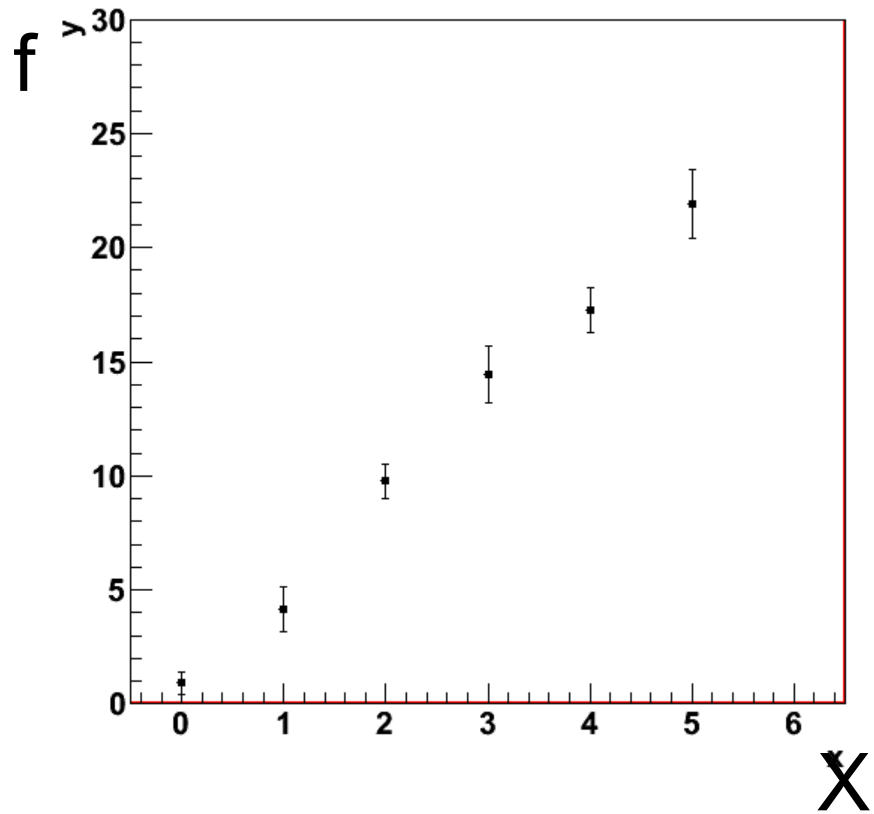
We already know the slope (e.g. from some theory)
Want to measure the offset b only.

$$\frac{\partial S}{\partial b} = -2\sum \frac{\left(f_i - a x_i - b\right)}{\sigma_{i,f}^2} = 0 \qquad S = \sum \frac{\left(f_i - a x_i - b\right)^2}{\sigma_{i,f}^2}$$

$$b = \frac{1}{\sum \frac{1}{\sigma_{i,f}^2}} \sum \frac{(f_i - a x_i)}{\sigma_{i,f}^2} \longrightarrow \sigma_b = \sqrt{\frac{1}{\sum \frac{1}{\sigma_{i,f}^2}}}$$

Example: Find the best straight line through the following measured points:

x	0	1	2	3	4	5
f	0.92	4.15	9.78	14.46	17.26	21.9
σ	0.5	1.0	0.75	1.25	1.0	1.5



Try to find the best fit line at home!

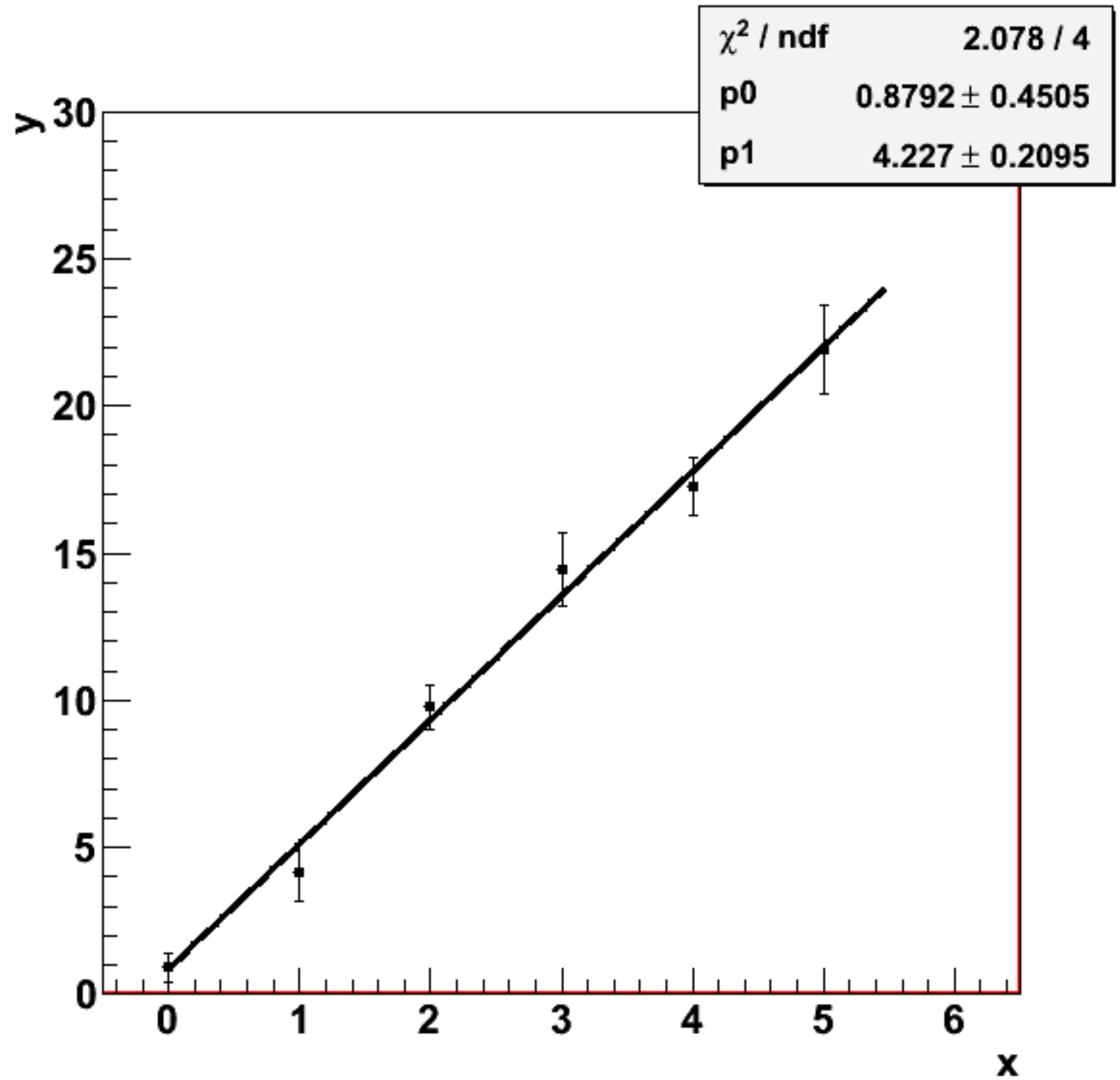
Being able to do this is needed in many labs!

see also http://skipper.physics.sunysb.edu/~joanna/Lectures/PHY-251-252/PHY251/PHY252-least-squares-example.pdf

Result:

a=4.227 b=0.879

$(\sigma_a)^2=0.044$ $(\sigma_b)^2=0.203$



Rounding:

- 2 significant digits

$$b = 0.88 \pm 0.45$$

$$a = 4.23 \pm 0.21$$

- 1 significant digit

$$b = 0.9 \pm 0.5$$

$$a = 4.2 \pm 0.2$$

Always round the fit results (a and b) and their uncertainties to the same significant digit

Can you reproduce those numbers using the derived formulas?